Insider Trading, Stochastic Liquidity and Equilibrium Prices☆

Pierre Collin-Dufresne

Carson Family Professor of Finance, Columbia University, and EPFL & SFI and NBER

Vyacheslav Fos

University of Illinois at Urbana-Champaign

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Abstract

We extend Kyle's (1985) model of insider trading to the case where liquidity provided by noise traders follows a general stochastic process. Even though the level of noise trading volatility is observable, in equilibrium, measured price impact is stochastic. If noise trading volatility is mean-reverting, then the equilibrium price follows a multivariate 'stochastic bridge' process, which displays stochastic volatility. This is because insiders choose to optimally wait to trade more aggressively when noise trading activity is higher. In equilibrium, market makers anticipate this, and adjust prices accordingly. More private information is revealed when volatility is higher. In time series, insiders trade more aggressively, when measured price impact is lower. Therefore, execution costs to uninformed traders can be higher when price impact is lower.

Keywords: Kyle model, insider trading, asymmetric information, liquidity, price impact, market depth, stochastic volatility, execution costs, continuous time.

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Email addresses: pc2415@columbia.edu (Pierre Collin-Dufresne), vfos@illinois.edu (Vyacheslav Fos)

1. Introduction

In his seminal contribution, Kyle (1985) derives the equilibrium price dynamics in a model where a large trader possesses long-lived private information about the value of a stock that will be revealed at some known date, and optimally trades into the stock to maximize his expected profits. Risk-neutral market makers try to infer from aggregate order flow the information possessed by the insider. Because order flow is also driven by uninformed 'noise traders,' who trade solely for liquidity purposes, prices are not fully revealing. Instead, prices respond linearly to order flow. Kyle's lambda, which measures the equilibrium price impact of order flow is constant in the model. In the cross-section, stocks with more insider trading relative to noise trading experience larger price impact. More precisely, Kyle's lambda, which can be estimated from a regression of price change on order flow, is higher for stocks with more private information (relative to the amount of liquidity trading).

Recent empirical results (Collin-Dufresne and Fos (CF 2012)) suggest however, that informed traders may condition their trades on such measures of price-impact, thus leading to selection biases in estimates of liquidity measures. Specifically, CF document that informed traders tend to trade more aggressively, when measures of stock price liquidity are better (and measured price impact is lower). In Kyle's model (and indeed in many extensions, such as Back, 1992; Foster and Viswanathan, 1990) this cannot happen because measures of market depth are constant in equilibrium (as discussed in Back and Pedersen, 1998).

In this paper, we propose an extension of Kyle's model that helps explain some of these findings. We generalize Kyle's model (in the continuous time formulation given by Back, 1992) to allow noise trading volatility to change stochastically over time. The main (economic) restriction we put on the process is that it be independent of the insider's private information and of shocks to the order flow.

We ask the following questions. How does the insider adapt his optimal trading strategy to account for these time-varying noise trader liquidity shocks? How are the equilibrium price dynamics affected by these shocks, which are by assumption orthogonal to the private information of the insider, and to the aggregate order flow dynamics?

Kyle's model provides the insight that the larger the noise trading liquidity for a given amount of private information, the more aggressive the insider will trade, since, in equilibrium, his optimal trading rule is inversely related to Kyle's lambda (a measure of price impact). The insider makes more profits when there is more noise trading, since the market maker can recoup more profits himself from the greater volume of uninformed traders. In a dynamic setting where 'noise trader volume' changes stochastically, one therefore expects price impact measures to move over time, and the insider to adjust his trading to take advantage of those moments when 'liquidity' is greater.

At first, this problem may seem like a simple extension of the Back (1992) model, as one might conjecture that one can simply 'paste together' Kyle economies with different levels of noise-trading volatility. However, this is not so. Indeed, the insider will optimally choose to trade less in the lower liquidity states than he would were these to last forever, because he anticipates the future opportunity to trade more when liquidity is better and he can reap a larger profit. Of course, in a rational expectations' equilibrium, the market maker foresees this, and adjusts prices accordingly.

Therefore, price dynamics are more complex than in the standard Kyle model.

First, in equilibrium, price impact measures are stochastic. Price impact is larger when noise-trading volatility is lower. When noise trading volatility is lower, the informed trader trades less aggressively. Thus, in equilibrium, the probability of facing an informed trader, may actually decrease in price impact (Kyle's lambda).

Second, market depth (the inverse of price impact) is a martingale in equilibrium, which implies that price impact (Kyle's lambda) is a submartingale, i.e., is expected to increase on average. This is in contrast to much of the previous theoretical models (e.g., Baruch, 2002; Back and Baruch, 2004; Back and Pedersen, 1998; Admati and Pfleiderer, 1988; Caldentey and Stacchetti, 2010).[1] The prediction of our model that price impact is expected to rise on average is consistent with the empirical evidence in Madhavan et al. (1997) who find that estimated execution costs rise significantly on average over the day.

Third, when noise trading volatility is predictable, then equilibrium price dynamics display time-varying volatility. Because, the insider has an incentive to wait and trade during higher 'liquidity' states, in equilibrium, the price follows a multi-variate 'stochastic bridge' process, whose volatility is stochastic and driven by both noise trader volatility and the posterior variance of the insider's private information. Mathematically, the price process resembles the Brownian Bridge process used, for example, in Back (1992). However, the dynamics are more complex in that they are multi-variate and exhibit stochastic volatility. These dynamics should also be useful for applications outside the current model. Interestingly, when noise trader volatility is a martingale (i.e., it is unforecastable), equilibrium prices are identical to the original Kyle/Back equilibrium, even-though market depth is stochastic.

Fourth, when noise trader volatility is predictable, then information revelation, as measured by the decrease in the posterior variance of the informed's signal, is faster when price volatility is higher. In particular, when there is more noise trading, more information gets into prices. This is consistent with the evidence in Foster and Viswanathan (1993) who find a positive relation between their estimates of the adverse

[1]In Baruch (2002) and Back and Baruch (2004) Kyle's lambda is actually a super-martingale. As discussed in these papers, this arises because the insider faces a random deadline or is risk-averse. See the discussion in Back and Pedersen (1998) page 387.

selection component of trading costs and volume (for actively traded stocks).

The model highlights the fact that, as often, 'liquidity,' has several facets. Empirical measures of price impact may not be good proxies for the level of 'adverse selection risk' (as measured by the arrival rate of private information) if the volatility of noise trading liquidity fluctuates over time. For example, in a regime switching model of noise trading volatility, execution costs for uninformed traders are lower in the low noise trading volatility regime than in the high noise trading volatility regime. However, the high noise trading volatility regime is also the one where measured price impact is lower.

This paper is primarily related to work by Back and Pedersen (1998) who extend Kyle's original model to allow for deterministically changing noise trader volatility to capture intra-day patterns (clustering) of liquidity trading. They also find that the informativeness of orders and the volatility of prices follow the same pattern as the liquidity trading. However, in their setting there are no systematic patterns in the price impact of orders. This implies that 'expected execution costs of liquidity traders do not depend on the timing of their trades' (BP p. 387). Instead, in our model, equilibrium price impact is a submartingale, i.e., is expected to increase on average. Expected execution costs of noise traders tend to be higher when noise trader volatility is lower.

Our paper is also related to a long list of papers investigating the impact of asymmetric information on asset prices and volatility (see Brunnermeier (2001) for a survey). For example, Admati and Pfleiderer (1988) investigate a dynamic economy, with myopic agents (essentially a sequence of one-period Kyle models), where they generate time variation in price volatility. In their model, price volatility is stochastic because the amount of private information changes from period to period, not because noise trading volatility is time varying.[2]

[2]Indeed, in the standard Kyle model, price volatility only depends on the volatility of private information and not on noise trading volatility. Since Admati and Pfleiderer (1988) consider a sequence

Foster and Viswanathan (1990) also propose a model with discrete jumps in noise trader volume. In their framework, market depth is also constant over time (as proved in Back and Pedersen, 1998).

As pointed out in Kyle and Vila (1991), noise trading liquidity (which one typically thinks of as a measure of market inefficiency) can incite traders with private information to act on their information, and thus, paradoxically, to make markets more efficient. We find, somewhat related, that more information (as measured by the decrease in private information posterior variance) gets into prices when price level volatility is higher, which tends to occur when the level of noise trading volatility is higher.

Section 2 introduces the general model and solves for an equilibrium. Section 3 investigates a few special cases, arbitrary deterministic noise trading volatility, and continuous time Markov Chain, to show some numerical simulations of equilibrium quantities. Section 4 concludes.

2. Informed Trading with Stochastic Liquidity Shocks

We extend Kyle's (1985) model (in the continuous time formulation given by Back, 1992) to allow for time varying volatility of noise trader liquidity shocks. As in Kyle, we assume there is an insider trading in the stock with perfect knowledge of the terminal value v. The insider is risk-neutral and maximizes the expectation of his terminal profit:

$$\max_{\theta_t} \mathrm{E}\left[\int_0^T (v - P_t)\theta_t dt \,|\mathcal{F}_t^Y, v\right], \tag{1}$$

of such one period models, where informed traders have short-lived private information, all variation in price volatility arises because of variation in the private information volatility. (Watanabe, 2008, extends their work to capture GARCH features in equilibrium prices, by directly incorporating stochastic volatility in the (short-lived) private information process.) This is very different from our model where the insider has long-lived information and optimally chooses to trade when noise trading volatility is high, thus generating a link between noise trading volatility and price volatility.

where we denote by $\mathcal{F}_t^Y$ the information filtration generated by observing the entire past history of aggregate order flow Y (which we denote by $Y^t = \{Y_s\}_{s\leq t}$).[3] In addition, the insider knows the actual value v of the stock, and, of course, his own trading. Following Back (1992) we assume that the insider chooses an absolutely continuous trading rule θ that satisfies $\mathrm{E}[\int_0^T \theta_s^2 ds] < \infty$.[4]

The market maker is also risk-neutral, but does not observe the terminal value. Instead, he has a prior that the value v is normally distributed $N(\mu_0, \Sigma_0)$.

The market maker only observes the aggregate order flow arrival:

$$dY_t = \theta_t dt + \sigma_t dZ_t, \tag{2}$$

where Z_t is a standard Brownian motion independent of v. We assume that the uninformed order flow volatility, σ_t, follows a general process, that is independent of the Brownian motion driving order flow. Specifically, we assume there is a Brownian motion M_t, which is independent both of v and of Z, such that:

$$d\sigma_t = m(t, \sigma^t)dt + \nu(t, \sigma^t)dM_t, \tag{3}$$

where the drift and diffusion of σ can depend on the past history of σ, but not on the

[3] In the standard Kyle-Back model, assuming that the informed observes total order flow is innocuous, since if the insider only observes equilibrium prices, he can typically recover the total order flow (and, given his own trading, the uninformed order flow). In our setting, when uniformed order flow has stochastic volatility, this assumption is important for our equilibrium construction. Alternatively, we could assume that the informed agent observes prices *and* uninformed order flow volatility. The point is that observing total order flow allows to recover uninformed order flow volatility, whereas observing only prices may not (we give some examples below, where equilibrium prices are independent of noise trading volatility, even though the insider's trades depend on it). The assumption that insiders observe noise-trader volatility, seems reasonable given that volume information is available in many markets.

[4] A shown in Back, it is optimal for the insider to choose an absolutely continuous trading strategy, since, in continuous time, the market maker can immediately infer from the quadratic variation of the order flow the informed component with infinite variation.

history of Y (or Z). Further, we assume they satisfy standard integrability requirements for the SDE to admit a unique strong solution and allow standard filtering tools to be used (specifically, the technical conditions imposed in theorem 12.1 p. 22 in Liptser and Shiryaev (LS 2001)). Importantly, we assume that the volatility process is uniformly bounded away from zero (to avoid the case of 'degenerate learning').[5] For simplicity we also assume that the volatility is bounded above uniformly.[6] Specifically, we assume there are two constants $\underline{\sigma}, \overline{\sigma}$ such that $0 < \underline{\sigma} \leq \sigma_t \leq \overline{\sigma}$.

We assume that both the market maker and the insider observe the history of σ perfectly. This is natural, since by observing aggregate order flow in continuous time, its quadratic variation is perfectly observed. Thus the filtration $\mathcal{F}_t^Y$ contains both histories of order flow (Y^t), and of volatility (σ^t).

We ask the following questions. How does the insider adapt his optimal trading strategy to account for these time-varying noise trader liquidity shocks? How is the equilibrium price dynamics affected by these shocks, which are by assumption orthogonal to the private information of the insider, and to the aggregate order flow dynamics?

At first, this problem may seem like a trivial extension of the Kyle (1985) model, as one might conjecture that one can simply 'paste together' Kyle economies with different levels of noise-trading volatility. However, this is not so. Indeed, the insider will optimally choose to trade less in the lower liquidity states than he would were these to last forever, because he anticipates the future opportunity to trade more when liquidity is better and he can reap a larger profit. Of course, in a rational expectations' equilibrium, the market maker foresees this, and adjusts prices accordingly.

To solve for an equilibrium, we proceed in a few steps. First, we derive the

[5]The positive lower bound is uniform across all (t, ω), see condition 11.6 page 2 in LS (2001).

[6]The assumptions are stronger than necessary, but simplify the derivation of the equilibrium in the general case. In the last section, we relax some of these assumptions. Specifically, we construct equilibria where M_t is not a purely continuous martingale, and where σ_t is not bounded uniformly.

dynamics of the stock price consistent with the market maker's risk-neutral filtering rule, conditional on a conjectured trading rule followed by the insider. Then we solve the insider's optimal portfolio choice problem, given the assumed dynamics of the equilibrium price. Finally, we show that the conjectured rule by the market maker is indeed consistent with the insider's optimal choice.

Since the market maker is risk-neutral, equilibrium imposes that

$$P_t = \mathrm{E}\left[v \,|\, \mathcal{F}_t^Y\right]. \tag{4}$$

We further, conjecture that the trading strategy of the insider will be linear in his per period profit, i.e.,

$$\theta_t = \beta(t, \sigma^t, \Sigma_t)(v - P_t), \tag{5}$$

where β measures the speed at which the insider decides to close the gap between the fundamental value v (known only to him) and the market price P_t and where we define Σ_t as the conditional variance of the terminal payoff:

$$\Sigma_t = \mathrm{E}\left[(v - P_t)^2 \,|\, \mathcal{F}_t^Y\right]. \tag{6}$$

Given our assumptions, this is a standard conditionally gaussian filtering problem, as treated in LS (2001) Chapter 12. We can prove the following result:

Lemma 1. *If the insider adopts a trading strategy of the form given in (5), then the stock price given by equation (4) satisfies* $P_0 = \mu_0$ *and:*

$$dP_t = \lambda(t, \sigma^t, \Sigma_t) dY_t, \tag{7}$$

where the price impact is a function of the conjectured trading rule:

$$\lambda(t, \sigma^t, \Sigma_t) = \frac{\beta(t, \sigma^t, \Sigma_t)\Sigma_t}{\sigma_t^2}. \tag{8}$$

Further, the dynamics of the posterior variance are given by:

$$d\Sigma_t = -\lambda(t, \sigma^t, \Sigma_t)^2 \sigma_t^2 dt. \tag{9}$$

Proof 1. *This follows directly from an application of theorems 12.6, 12.7 in LS 2001. We provide a simple 'heuristic' motivation of the result using standard Gaussian projection theorem below.*

$$\begin{aligned}
P_{t+dt} &= \mathrm{E}\left[v \,|\, Y^t, Y_{t+dt}, \sigma^t, \sigma_{t+dt}\right] &(10)\\
&= \mathrm{E}\left[v \,|\, Y^t, \sigma^t\right] + \frac{Cov(v, Y_{t+dt} - Y_t \,|\, Y^t, \sigma^t)}{V(Y_{t+dt} - Y_t \,|\, Y^t, \sigma^t)}(Y_{t+dt} - Y_t - \mathrm{E}[Y_{t+dt} - Y_t \,|\, Y^t, \sigma^t]) &(11)\\
&= P_t + \frac{\beta\Sigma_t dt}{\beta^2\Sigma_t dt^2 + \sigma_t^2 dt}(Y_{t+dt} - Y_t) &(12)\\
&\approx P_t + \frac{\beta\Sigma_t}{\sigma_t^2} dY_t. &(13)
\end{aligned}$$

The second line uses the fact that the dynamics of σ_t is independent of the asset value distribution and of the innovation in order flow. The third line uses the fact that the expected change in order flow is zero for the conjectured policy. The last line follows from going to the continuous time limit (with $dt^2 \approx 0$). Similarly, by the projection theorem, we have:

$$Var\left[v \,|\, Y^t, Y_{t+dt}, \sigma^t, \sigma_{t+dt}\right] = Var\left[v \,|\, Y^t, \sigma^t\right] - (\frac{\beta\Sigma_t}{\sigma_t})^2 Var\left[Y_{t+dt} - Y_t \,|\, Y^t, \sigma^t\right], \tag{14}$$

which gives:

$$\Sigma_{t+dt} = \Sigma_t - \lambda_t^2 \sigma_t^2 dt. \tag{15}$$

In the spirit of a rational expectation's equilibrium, we now conjecture a price impact function of the form:

$$\lambda(t,\sigma^t,\Sigma_t)=\sqrt{\frac{\Sigma_t}{G_t}}, \tag{16}$$

where G_t solves the following recursive stochastic differential equation:

$$\sqrt{G_t}=\mathrm{E}\left[\int_t^T \frac{\sigma_s^2}{2\sqrt{G_s}}ds|\sigma^t\right]. \tag{17}$$

We can show the following lemma

Lemma 2. *There exists a maximal bounded solution G_t to the recursive equation (17). Further, that solution satisfies*

$$\underline{\sigma}^2\,(T-t)\leq G_t\leq \overline{\sigma}^2\,(T-t) \tag{18}$$

Proof 2. *We note that $y_t=\sqrt{G_t}$ solves the Backward stochastic differential equation*

$$dy_t=-f(t,y_t)dt-\Lambda_t dM_t$$

with $f(t,y_t)=\frac{\sigma_t^2}{2y_t}$ and with terminal condition $y_T=0$. Now $f(t,y_t)\leq \ell(y_t)\ \forall(t,\omega)$ where we define the function $\ell(y)=\frac{\overline{\sigma}^2}{2y}$. We can thus compute $\int_0^\infty \frac{dx}{\ell(x)}=\int_{-\infty}^0 \frac{dx}{\ell(x)}=\infty$. Thus $\ell(x)$ is super-linear as shown in lemma 1 of Lepeltier and San Martin (1997). Their theorem 1 then applies, which gives us the existence of a maximal bounded solution for y_t (and therefore for G_t).

Now consider the solution to the following Backward equation

$$dx_t=-\frac{\underline{\sigma}^2}{2x_t}dt-\tilde{\Lambda}_t dM_t$$

with terminal condition $x_T=0$. It can be computed straightforwardly as $x_t=\underline{\sigma}\sqrt{T-t}$ (note $\tilde{\Lambda}_t=0$). Since $\forall(t,\omega);\ \ f(t,y)\geq\frac{\underline{\sigma}^2}{2y}$ we can use the comparison result Corollary 2 *of Lepeltier and San Martin (1997) to obtain*

$$y_t \geq x_t \;\; \forall(t, \omega) \tag{19}$$

which gives the lower bound on the maximal solution for G_t. The upper bound follows immediately from a slightly more general bound on G that we derive in lemma 5 below (see equation (70)).

Remark 1. *The theorem in Lepeltier and San Martin (1997) does not guarantee the uniqueness of the solution to the BSDE. However, the use of the maximal solution in the construction of the equilibrium seems sensible since it achieves the highest value function for the insider, as we show below.*

We assume that the insider takes the price dynamics given by equations (7), (9), (16), and (17) as given to solve his 'partial equilibrium' problem:

$$J(t) = \max_{\theta_t} \mathrm{E}\left[\int_0^T (v - P_t)\theta_t dt \,|\, v, \mathcal{F}_t^Y\right]. \tag{20}$$

Then, we will verify that his optimal trading strategy is indeed of the form conjectured in (5).

We first prove an important property of our conjectured equilibrium price process. Namely that it converges almost surely to the value v, known (only) to the insider, at maturity T. This guarantees that all private information will have been incorporated in equilibrium prices at maturity. This property is analogous to the result proved in Back (1992), that equilibrium prices in the continuous time Kyle model follow a standard Brownian Bridge. In our framework, equilibrium prices follow a kind of 'stochastic bridge' process, in that the price converges to the known value, but the process may display stochastic volatility due to the stochastic noise trader volatility shocks.

Theorem 1. *Suppose price dynamics are given by equations (7), (9),(16), and (17), then the price process P_t converges in $\mathcal{L}^2$ to v at time T.*

Proof 3. *The conjectured equilibrium price process is:*

$$dP_t = \frac{(v - P_t)}{G_t}\sigma_t^2 dt + \sqrt{\frac{\Sigma_t}{G_t}}\sigma_t dZ_t \tag{21}$$

$$d\Sigma_t = -\frac{\Sigma_t}{G_t}\sigma_t^2 dt. \tag{22}$$

To prove that $P_T = v$ in $\mathcal{L}^2$, we consider the process $X(t) = P_t - v$ and show that $\lim_{t\to T} \mathrm{E}[X(t)] = 0$ *and that* $\lim_{t\to T} \mathrm{E}[X(t)^2] = 0$. *This establishes convergence in $\mathcal{L}^2$.*

Note that

$$X(t) = e^{-\int_0^t \frac{\sigma_u^2}{G_u}du} X_0 + \int_0^t e^{-\int_s^t \frac{\sigma_u^2}{G_u}du}\sqrt{\frac{\Sigma_s}{G_s}}\sigma_s\, dZ_s \tag{23}$$

$$:= I_1(t) + I_2(t), \tag{24}$$

where the second line defines the integrals I_1, I_2. Equation (18) implies that

$$-\int_0^t \frac{\sigma_u^2}{G_u}du \leq \frac{\underline{\sigma}^2}{\overline{\sigma}^2}\log(\frac{T-t}{T})$$

It follows immediately from this inequality that

$$\lim_{t\to T} \mathrm{E}[I_1(t)^2] = \lim_{t\to T} \mathrm{E}[I_1(t)] = 0. \tag{25}$$

Further, note that:

$$\mathrm{E}\left[\int_0^t \left\{e^{-\int_s^t \frac{\sigma_u^2}{G_u}du}\sqrt{\frac{\Sigma_s}{G_s}}\sigma_s\right\}^2 ds\right] = \mathrm{E}\left[-\int_0^t e^{-\int_s^t 2\frac{\sigma_u^2}{G_u}du} d\Sigma_s\right]. \tag{26}$$

Using integration by parts, we find:

$$\mathrm{E}\left[\int_0^t \left\{ e^{-\int_s^t \frac{\sigma_u^2}{G_u} du} \sqrt{\frac{\Sigma_s}{G_s}} \sigma_s \right\}^2 ds \right] = \mathrm{E}\left[\Sigma_t - e^{-\int_0^t 2\frac{\sigma_u^2}{G_u} du} \Sigma_0 \right]. \tag{27}$$

We also have $\Sigma_t = \Sigma_0 e^{-\int_0^t \frac{\sigma_u^2}{G_u} du}$. *Thus we conclude that the stochastic integral* $I_2(t)$ *has finite second moment. It therefore is a martingale (thus* $\mathrm{E}[I_2(t)] = 0$*) and by the Itô Isometry we have:*

$$\mathrm{E}[I_2(t)^2] = \mathrm{E}\left[\Sigma_0 e^{-\int_0^t \frac{\sigma_u^2}{G_u} du} (1 - e^{-\int_0^t \frac{\sigma_u^2}{G_u} du})\right]. \tag{28}$$

It follows that $\lim_{t \to T} \mathrm{E}[I_2(t)^2] = 0$ *and that* $\lim_{t \to T} \mathrm{E}[X(t)] = \lim_{t \to T} \mathrm{E}[X(t)^2] = 0$. *This establishes* $\mathcal{L}^2$ *convergence.*

We now establish an interesting property of the conjectured price process. This contrasts our framework from much of the previous literature. In the original Kyle model price impact is constant. In extensions of that model (Back, 1992; Back and Pedersen, 1998; Baruch, 2002; Back and Baruch, 2004), price impact is either a martingale, or a super-martingale. In these models, price impact measures have to improve (i.e, decrease) on average over time, to incite the insider to not trade too aggressively initially.[7] Instead, our framework is the first (to our knowledge) where price impact measures are expected to deteriorate (i.e, increase) over time. Indeed, we find:

Lemma 3. *Market depth (which is the inverse of the price impact, i.e, Kyle's lambda) is a martingale that is orthogonal to the aggregate order flow. It follows that price impact (Kyle's lambda) is a submartingale.*

[7]Motives to trade more aggressively early on are due to risk-aversion and a random exogenous deadline. It would be interesting to combine risk-aversion or random deadline, with stochastic noise trader volatility. It is likely that price impact would be neither a sub nor a super martingale in that case.

Proof 4. *Note that from its definition the G_t process satisfies:*

$$d\sqrt{G_t} + \frac{\sigma_t^2}{2\sqrt{G_t}}dt = d\mathcal{M}_t, \tag{29}$$

where $\mathcal{M}_t = \mathrm{E}[\int_0^T \frac{\sigma_t^2}{2\sqrt{G_t}}dt \,|\, \sigma^t]$ is a martingale (adapted to the filtration generated by the noise-trader volatility process) by the law of iterated expectation. It follows, by definition of the process σ_t, that $d\mathcal{M}_t dZ_t = 0$.

From its definition in (16) and the definition for Σ and G above we obtain:

$$\begin{aligned} d\frac{1}{\lambda(t)} &= \frac{1}{\sqrt{\Sigma_t}}d\sqrt{G_t} - \frac{\sqrt{G_t}}{2(\Sigma_t)^{3/2}}d\Sigma_t & (30)\\ &= \frac{1}{\sqrt{\Sigma_t}}d\mathcal{M}_t. & (31)\end{aligned}$$

It also follows that $d\frac{1}{\lambda_t}dY_t = 0$.

To prove that λ is a submartingale we apply Jensen's inequality. We have: $\frac{1}{\lambda_t} = \mathrm{E}_t[\frac{1}{\lambda_s}] \geq \frac{1}{\mathrm{E}_t[\lambda_s]}$. It follows that $\lambda_t \leq E_t[\lambda_s]$.

We now prove the following result, which establishes the optimal trading rule for the insider:

Theorem 2. *Suppose price dynamics are given by equations (7), (9),(16), and (17), and that the drift $m(t,\sigma^t)$ and volatility $\nu(t,\sigma^t)$ of the σ_t process are such that the following technical conditions are satisfied:*

A1 The local martingales $\int_0^u (v - P_t)\sqrt{\frac{\Sigma_t}{G_t}}\sigma_t dZ_t$ and $\int_0^u \frac{(v-P_t)^2+\Sigma_t}{2\sqrt{\Sigma_t}}d\mathcal{M}_t$ are 'true' martingales.

A2 $\lambda_T = \sqrt{\frac{\Sigma_T}{G_T}} > 0$ a.s.

Then the optimal value function is given by:

$$J(t) = \frac{(v-P_t)^2 + \Sigma_t}{2\lambda_t}. \tag{32}$$

The optimal strategy is given by:

$$\theta_t^* = \frac{1}{\lambda_t}\frac{\sigma_t^2}{G_t}(v - P_t). \tag{33}$$

Proof 5. *Apply Itô's rule to the conjectured value function to get*

$$dJ(t) = \frac{(v-P_t)^2 + \Sigma_t}{2} d\frac{1}{\lambda_t} + \frac{1}{\lambda_t}\left(-(v-P_t)dP_t + \frac{1}{2}dP_t^2\right) - (v-P_t)dP_t\, d\frac{1}{\lambda_t} + \frac{1}{2\lambda_t}d\Sigma_t. \tag{34}$$

The insider takes the price impact process as given and assumes the price process follows:

$$dP_t = \lambda_t\left(\theta_t dt + \sigma_t dZ_t\right),$$

with the lambda process as in equation (16) above. Using lemma 3 and the Σ_t *dynamics, and integrating the above we obtain:*

$$J(T) - J(0) + \int_0^T (v-P_t)\theta_t dt = \int_0^T (v-P_t)\sqrt{\frac{\Sigma_t}{G_t}}\sigma_t dZ_t + \int_0^T \frac{(v-P_t)^2 + \Sigma_t}{2\sqrt{\Sigma_t}} d\mathcal{M}_t. \tag{35}$$

Now, since $J(T) \geq 0$ *it follows by taking expectation and using A1 that*

$$\mathrm{E}\left[\int_0^T (v-P_t)\theta_t dt\right] \leq J(0) \tag{36}$$

for any admissible policy $\{\theta_t\}$*. Further, if there exists a trading strategy* θ_t *consistent with the updating equations (8), such that* P_T *converges to* v *in* $\mathcal{L}^2$ *then, using A2, we have* $\mathrm{E}[J(T)] = 0$ *and the inequality holds with equality. As we have shown in theorem 1, the candidate policy* θ_t^* *proposed in the theorem indeed satisfies this. We have therefore proved the optimality of the value function and of the proposed policy.*

Remark 2. *The technical assumptions A1 and A2 are a bit unsatisfactory since they are about endogenous quantities. However, at this level of generality and specifically, without making further assumptions about* $m(t,\sigma^t)$*,* $\nu(t,\sigma^t)$*, it is difficult for us to give a more precise statement. In the next section however, we give six examples of the* σ_t

process where we can verify explicitly that these conditions hold. Note that in the original Kyle-Back model where σ_t is constant, these conditions are clearly satisfied.

The following result is an immediate consequence of theorems 1 and 2:

Theorem 3. *Under the conditions of theorem 2, there exists an equilibrium where the price process follows dynamics given in equations (7), (9),(16), and (17), and where the insider follows trading strategy established in theorem 2. The price impact (Kyle's lambda) follows a stochastic process, given in equation (16).*

It is interesting to compare our results to those of Kyle (1985), and Back and Pedersen (1998).

First, we see that the optimal trading strategy for the insider is to trade proportionally to the under-valuation of the asset $(v - P_t)$ at a rate that is inversely related to her price impact (λ_t) and to the remaining amount of noise trader risk as measured by the new equilibrium quantity $(\frac{G_t}{\sigma_t})$. The latter quantity reduces to the remaining time horizon $T - t$ in the original Kyle model when σ_t is constant.

Second, our expression for the price impact generalizes both BP's result (page 395) obtained for deterministically changing noise trader volatility and Kyle's result that price impact (the inverse of market depth) is proportional to the amount of private information that has not yet been incorporated into prices and inversely proportional to the amount of noise trading. Interestingly, the measure of the relevant noise amount we obtain, when noise trading is stochastic, is quite different from what obtains in the deterministic case, where it is simply the remaining total variance. In fact we show below (see lemma 5) that $G(t)$ is always smaller than the expected remaining noise-trader variance.

Lastly, our results shed more light on the dynamics of price impact. For example, BP find that, when noise trader volatility is deterministic but information arrives stochastically to the insider, then λ_t is a martingale. Instead, we find that with fixed information of the insider, but stochastically arriving noise trader volatility, the inverse of λ_t (which measures market depth) is a martingale. In both cases, the martingale condition reflects the fact that, in equilibrium, informed trading responds optimally to changes in the environment (information arrival in BP, noise trading in our case). Interestingly, in BP's model, because noise trading changes deterministically, the sensitivity of prices to orders does not change systematically over time. Instead, in our model, it does, since as shown in lemma 3 above, price impact is a submartingale. In other words, not only is price impact systematically lower when noise trading volatility is higher, but also on average we expect price impact to increase over time. Unlike in BP, we find that in equilibrium market depth can vary systematically over time. The information content of orders is not constant in our model.

How large are the profits to the insider?

Total unconditional profits of the informed in our model can be computed by integrating the value function over the unconditional prior distribution of v, as $\mathrm{E}^v[J(0)] = \frac{\Sigma_0}{\lambda_0} = \sqrt{\Sigma_0 G_0}$. Clearly, the profits depend on how much private information remains to be released to the market, and the total expected amount of noise trading as measured by the solution to the BDSE for G_0. Since the latter depends on the current state of liquidity, it is clear that the total profits generated by the insider will be path dependent, and a function of the realized noise-trading volatility. The more time the market spends in the higher liquidity state, intuitively, the higher the profits the insider will generate.

How does information get into prices?

Note that in equilibrium

$$d\Sigma_t = -dP_t^2, \tag{37}$$

which shows that information arrives at higher rate, when stock price volatility is high. As we show below, when noise trading volatility exhibits mean-reversion, then stock price volatility is stochastic and tends to be higher when noise trading volatility is higher. So paradoxically, the adverse selection cost (as measured by the probability of facing an informed trader) tends to be higher, when noise trading liquidity is higher. This is very different from the standard Kyle model, where private information decays at a constant rate that is independent of the noise trading volatility level.

What can we say about the execution costs for uninformed noise traders?

We define the aggregate execution (or slippage) costs incurred by liquidity traders at time T (defined pathwise) as:

$$\int_0^T \sigma_t dZ_t dP_t = \int_0^T \lambda_t \sigma_t^2 dt. \tag{38}$$

Intuitively, the total losses incurred between 0 and T by uninformed can be computed pathwise as:

$$\int_0^T (P_{t+dt} - v)\sigma_t dZ_t = \int_0^T (P_t + dP_t - v)\sigma_t dZ_t = \int_0^T \lambda_t \sigma_t^2 dt + \int_0^T (P_t - v)\sigma_t dZ_t. \tag{39}$$

The first component is the pure execution or slippage cost due to the fact that, in Kyle's model, agents submit market orders at time t that get executed at date $t+dt$ at a price set by competitive market makers. The second component is the pure fundamental loss due to the fact that based on the price they observe at t uninformed purchase a security with fundamental value v that is unknown to them. Note that since prices are set efficiently by market makers, on average this second component has zero mean. Therefore we obtain the result that, the unconditional expected total losses incurred by uninformed are equal to the unconditional expected execution costs incurred by uninformed. Further, as the next lemma shows, these are also equal the total unconditional expected profits of the insider. (Note, however, that pathwise neither quantity need be equal.)

Lemma 4. *Unconditional expected execution costs paid by the uninformed are equal to the unconditional expected profits of the insider:*

$$\mathrm{E}^v[\int_0^T \theta_t^*(v - P_t)dt] = \mathrm{E}^v[\int_0^T \sigma_t^2 \lambda_t dt] = \sqrt{\Sigma_0 G_0}. \tag{40}$$

Proof 6. *The insider's unconditional expected profits are*

$$\begin{aligned}
\mathrm{E}^v[\int_0^T \theta_t(v - P_t)dt] &= \mathrm{E}^v[\int_0^T \frac{\sigma_t^2}{\sqrt{\Sigma_t G_t}}(v - P_t)^2 dt] && (41)\\
&= \mathrm{E}^v[\int_0^T \frac{\sigma_t^2}{\sqrt{\Sigma_t G_t}}\Sigma_t dt] && (42)\\
&= \mathrm{E}^v[\int_0^T \sigma_t^2 \lambda_t dt], && (43)
\end{aligned}$$

where the first equality follows from the definition of θ^* *and the second from the law of iterated expectations. This is the same expression obtained for the execution costs paid by the uninformed. By definition this is also equal to* $\mathrm{E}^v[J(0)]$ *where the expectation superscript emphasizes that it is taken over the unconditional distribution of* v*. This gives the result.*

We will show numerically below, that aggregate execution costs to noise traders can be larger when noise trading liquidity is larger (even though in those states, price impact measured by lambda is lower). This is consistent with trading profits of the insider being larger when noise trading volatility is higher.

In the following section we consider a few specific examples of noise trader volatility process to illustrate the above result.

3. Examples

As is clear from the above proposition, much of the characterization of the equilibrium depends on the dynamics of the λ_t process, which, in turn, depends on the G_t and Σ_t processes. G_t solves a backward stochastic differential equation, which can be solved for specific choices of the noise trader volatility dynamics. In this section we consider a few special cases, for which we can characterize the equilibrium further.

3.1. Arbitrary deterministic volatility process

Suppose that the noise trader volatility follows an arbitrary deterministic volatility, then the backward equation for $G(t)$ simplifies to:

$$\sqrt{G(t)} = \int_t^T \frac{\sigma(u)^2}{2\sqrt{G(u)}} du. \tag{44}$$

In that case we can show the following.

Theorem 4. *A solution to equation (44) is the function* $G(t) = \int_t^T \sigma(u)^2 du$. *In that case, the equilibrium is identical to that derived in Kyle/Back, up to a deterministic time change given by* $\tau_t = \int_0^t \sigma_u^2 du$. *Indeed, private information is incorporated into prices at the same rate at which the amount of noise trader volatility decays. Price impact (and market depth) are constant. Kyle's lambda is given by:* $\lambda = \sqrt{\frac{\Sigma_0}{\tau_T}}$. *The optimal strategy of the insider is:* $\theta_t^* = \frac{\sigma_t^2}{\lambda \int_t^T \sigma_u^2 du}(v - P_t)$. *The equilibrium price process follows a time-changed Brownian bridge process:*

$$dP_t = \frac{(v - P_t)}{\tau_T - \tau_t} d\tau_t + \lambda dZ(\tau_t). \tag{45}$$

Proof 7. *Differentiating both sides of (44) we see that a solution satisfies* $G'(t) = -\sigma(t)^2$ *with boundary* $G(T) = 0$. *Thus we obtain the solution for* $G(t)$. *Further, we have* *from (9) with the definition of* $\lambda_t = \sqrt{\frac{\Sigma_t}{G_t}}$ *that:*

$$\frac{d\Sigma_t}{\Sigma_t} = \frac{dG(t)}{G(t)}. \tag{46}$$

This implies that $\frac{\Sigma_t}{G_t} = \frac{\Sigma_0}{G_0}$*, and hence that* λ *is constant. The optimal portfolio policy and the equilibrium price process follow from equations (7) and equation (33). Comparing the equation for the price to the Brownian bridge process obtained in Back (1992), it is clear that the dynamics are identical up to a deterministic time change given by* $\tau_t = \int_0^t \sigma_u^2 du$*.*

This result is consistent with the analysis in Back and Pedersen (1998) and specializes to the continuous time Kyle-model also derived in Back (1992) if $\sigma(t) = \sigma_u$ is constant (in which case $\lambda = \frac{\sigma_v}{\sigma_u}$ with $\sigma_v^2 = \frac{\Sigma_0}{T}$ is the annualized variance of the market maker's prior estimate of the asset value). Note that, since λ is constant in this example, it is trivially both a martingale and a submartingale. However, we give examples below where the lambda process is stochastic.

3.2. Positive martingale dynamics

Suppose that volatility follows an arbitrary (strictly positive) martingale process:

$$\frac{d\sigma_t}{\sigma_t} = \nu(t, \sigma^t) dM_t, \tag{47}$$

where as before M is a martingale (in fact, M_t need not even be a continuous Martingale) independent of Z. The backward equation for $G(t)$ is:

$$\sqrt{G(t)} = \mathrm{E}_t \left[\int_t^T \frac{\sigma_u^2}{2\sqrt{G(u)}} du \right]. \tag{48}$$

Then we can show the following.

Theorem 5. *A solution to equation (48) is* $G(t) = \sigma_t^2 (T - t)$*. In that case, private* *information is incorporated into prices linearly (independent of the level of noise trader volatility). Market depth (the inverse of Kyle's lambda) is proportional to noise trader volatility:*

$$\frac{1}{\lambda_t} = \frac{\sigma_t}{\sigma_v}, \tag{49}$$

where $\sigma_v^2 = \frac{\Sigma_0}{T}$ is the annualized initial private information variance level. The trading strategy of the insiders is:

$$\theta_t = \frac{\sigma_t}{\sigma_v(T-t)}(v - P_t). \tag{50}$$

Equilibrium price dynamics are identical to the original Kyle (1985) model:

$$dP_t = \frac{(v - P_t)}{T-t}dt + \sigma_v dZ_t. \tag{51}$$

In particular, stock price volatility is constant. The unconditional expected profit level of the insider at time zero is $T\sigma_v\sigma_0$.

Proof 8. *Plugging the guessed solution for G on the right hand side of equation (48) and using the martingale property of $\sigma(u)$ we obtain:*

$$\begin{aligned} \mathrm{E}_t\left[\int_t^T \frac{\sigma_u^2}{2\sqrt{G(u)}}du\right] &= \sigma(t)\int_t^T \frac{1}{2\sqrt{(T-u)}}du] && (52)\\ &= \sigma(t)\sqrt{(T-t)} && (53)\\ &= \sqrt{G(t)}, && (54) \end{aligned}$$

which confirms our guess. Further, we have from (9) and using the solution for $G(t)$ we obtain:

$$\frac{d\Sigma_t}{\Sigma_t} = -\frac{1}{T-t}. \tag{55}$$

The solution is:

$$\Sigma_t = \sigma_v(T-t). \tag{56}$$

Further, from its definition $\lambda_t = \sqrt{\frac{\Sigma_t}{G_t}} = \frac{\sigma_v}{\sigma(t)}$. From equation (7) we find the stock price *dynamics.*

This example shows that many of the features of Kyle's equilibrium survive when noise trader volatility follows *arbitrary* martingale dynamics. Indeed, we see that when noise trader volatility is not forecastable, private information gets into prices at the same rate as in the original economy (i.e., linearly). The equilibrium looks identical to the original Kyle-Back model where one substitutes a stochastic process σ_t for the constant volatility of uninformed order flow in the original model. Since in the original model, the equilibrium price process and the rate at which private information is revealed are independent of the volatility of uninformed order-flow, they are unchanged in this case. However, both the trading strategy of the insider and the price impact (Kyle's lambda) change. Both become stochastic. The insider trades more aggressively when uninformed order flow volatility is higher, but price impact moves in the exact opposite direction so that both effects cancel, leaving equilibrium prices unchanged. In equilibrium then, insiders cannot gain from timing their trades and thus their unconditional expected profit level is unchanged relative to what it would be in the Kyle-Back model with noise trader volatility set to a constant σ_0. Interestingly, even in this model however, price impact measures are stochastic and vary inversely with the level of noise trader volatility. Since the latter is a martingale, we see that on average, price impact is expected to deteriorate in this case.

This simple framework also suggests that price dynamics will become more complex if the level of noise trading volatility is predictable. We consider four such examples next. First, we consider the case where noise trader volatility grows (or decreases) at a constant rate. Second, we consider the case of general diffusion dynamics. Third, we consider a case of diffusion dynamics with mean-reversion. Fourth, we consider a regime switching model with state dependent predictability.

3.3. Constant growth rate

Suppose that volatility follows a geometric brownian motion process:

$$\frac{d\sigma_t}{\sigma_t} = mdt + \nu dW_t, \tag{57}$$

where W is a standard Brownian motion independent of Z and for simplicity we assume that m, ν are constant. The backward equation for $G(t)$ is as before. We can show the following.

Theorem 6. *A solution to equation (48) is the function* $G(t) = \sigma_t^2 B_t$ *where* $B_t = \frac{e^{2m(T-t)}-1}{2m}$. *In that case, private information is initially incorporated into prices at a faster (slower) rate than in the original Kyle model if* m *is negative (positive):*

$$\frac{\Sigma_t}{\Sigma_0} = \frac{1 - e^{-2m(T-t)}}{1 - e^{-2mT}}. \tag{58}$$

Market depth is given by:

$$\frac{1}{\lambda_t} = e^{-mt}\sigma_t\sqrt{\frac{B_0}{\Sigma_0}}. \tag{59}$$

The trading strategy of the insider is:

$$\theta_t = \frac{\sigma_t}{e^{-m(T-t)}B_t}\sqrt{\frac{B_0}{\Sigma_0}}(v - P_t). \tag{60}$$

Stock price dynamics are given by:

$$dP_t = \frac{(v - P_t)}{B_t}dt + e^{mt}\sqrt{\frac{\Sigma_0}{B_0}}dZ_t. \tag{61}$$

In particular, stock price volatility is a deterministic exponentially increasing (decreasing) function of time if noise trader volatility is expected to increase (decrease). The unconditional expected profit at time zero of the insider is $T\sigma_v\sigma_0\sqrt{\frac{B_0}{T}}$.

Proof 9. *Plugging the guessed solution for* G *on the right hand side of equation (17)* *and using the fact that $e^{-mu}\sigma(u)$ is a martingale, we obtain:*

$$\mathrm{E}\left[\int_t^T \frac{\sigma_u^2}{2\sqrt{G(u)}}du\right] = \sigma(t)\int_t^T \frac{e^{m(u-t)}}{2\sqrt{\frac{e^{2m(T-u)}-1}{2m}}}du \tag{62}$$

$$= \sigma_t\sqrt{\frac{e^{2m(T-t)}-1}{2m}}, \tag{63}$$

which confirms our guess.

Further, using (9) and the solution for $G(t)$ we obtain:

$$\frac{d\Sigma_t}{\Sigma_t} = -\frac{2m}{e^{2m(T-t)}-1}dt. \tag{64}$$

The solution is:

$$\frac{\Sigma_t}{\Sigma_0} = \frac{1-e^{-2m(T-t)}}{1-e^{-2mT}}. \tag{65}$$

Further, from its definition:

$$\lambda_t = \sqrt{\frac{\Sigma_t}{G_t}} = \sqrt{\frac{\Sigma_0}{G_0}}\frac{\sigma_0}{e^{-mt}\sigma_t}. \tag{66}$$

Equation (7) gives the stock price dynamics.

This example shows that as soon as there is some predictability, then the equilibrium differs from the standard Kyle-Back solution. For example, if noise trader volatility is expected to increase ($m > 0$), then the insider has an incentive to trade more aggressively (for a given level of noise trader volatility and of expected profit $v - P$), because an increase in m raises the amount of remaining cumulative noise trading ($G(t)$) at all times. His unconditional expected profit is increasing in m (because G_0 is increasing in m). We plot in figure 1 the optimal trading strategy of the insider normalized by its expected profit level ($\theta_t/(v-P_t)$) for different levels of m and in figure 2 the corresponding $G(t)$ function, holding the noise trader volatility fixed at $\sigma_t = 0.4$. Interestingly, this results in private information getting revealed at a different speed than in the benchmark economy, where volatility is unforecastable (or constant).

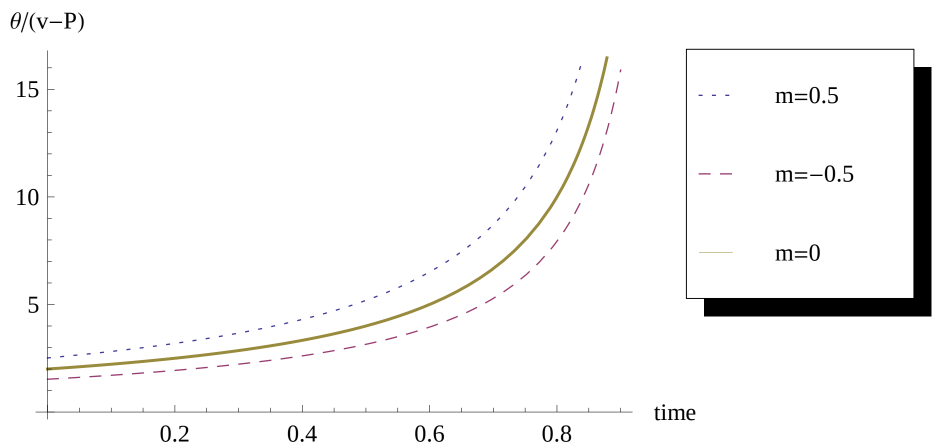

Figure 1: Trading strategy of the insider normalized by his expected profit $(\theta_t/(v - P_t))$ for a given fixed level of noise trader volatility plotted against time and for different levels of expected growth rate of noise trader volatility.

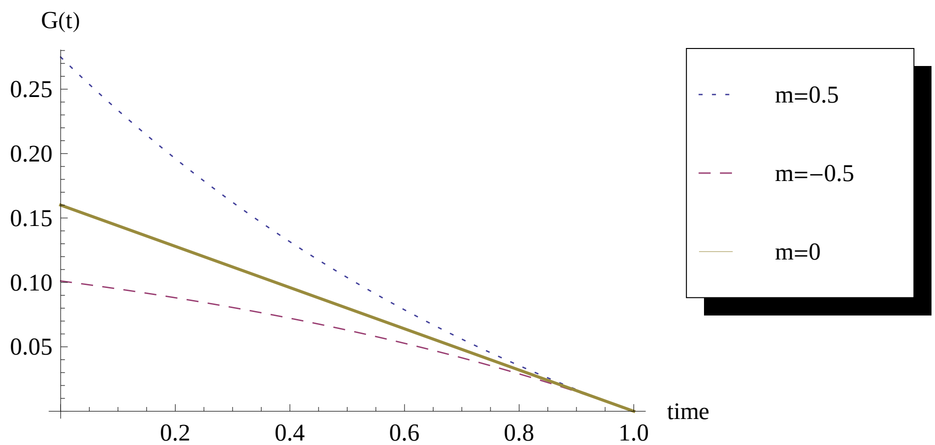

Figure 2: Expected remaining cumulative noise trading variance $(G(t))$ plotted against time and for different levels of expected growth rate of noise trader volatility.

Figure 3 plots the path of the posterior variance of the private information signal for three cases $m = 0.5$, $m = 0$ and $m = -0.5$. It is remarkable that private information is revealed following a deterministic path, which only depends on the expected rate of change in noise trading volatility, despite the fact that the strategy of the insider is

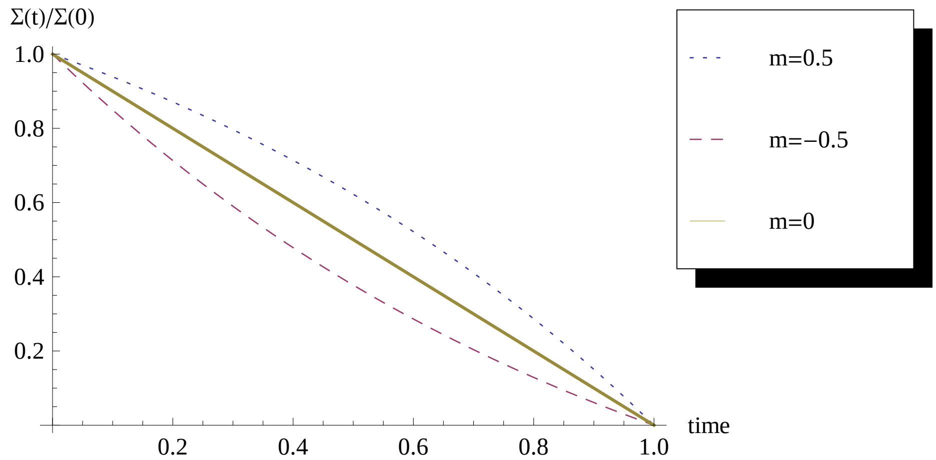

Figure 3: Path of posterior variance of the insider's private information Σ_t for various values of the expected change in noise trader volatility m.

stochastic. This is of course the result of the offsetting effect noise trading volatility has on the price impact coefficient λ_t. If the level of noise trading variance changes, the insider trades more or less aggressively, but price impact changes one for one, making price dynamics and information revelation independent of the volatility level. If variance is expected to increase, then private information gets into prices more slowly initially, and then faster when the insider trades more aggressively. So posterior variance follows a deterministic concave path if noise trader volatility is expected to increase, but a convex path if it is expected to decrease. As a result, the equilibrium price process exhibits time varying volatility. Its volatility increases (decreases) exponentially if noise trader volatility is expected to increase (decrease). Interestingly, price volatility is deterministic, despite stochastic noise trader volatility and stochastic market depth. From a mathematical point, the price process follows a one-factor Markov stochastic bridge process with non-time homogeneous volatility (recall that P_t almost surely converges to v at time T).

In this economy, we obtain an interesting separation result. The strategy of the insider (θ_t) and the price impact measures (λ_t) expressed as a function of the level of noise trader volatility (σ_t) are independent of the volatility of the noise trader volatility (ν). As a result, the informational efficiency of prices, the price process, and the unconditional expected profits of the insider only depend on initial conditions (Σ_0, σ_0) and the expected growth rate of noise trader volatility (m).

However, the time-series dynamics of the price impact measure and the optimal strategy of the insider is stochastic and varies with σ_t, which of course depends on ν. An implication is that we may see lots of variation in estimates of price impact measures (Kyle's lambda) in time series (i.e, at different times along one path) and in cross-section (i.e., at the same time across different 'economies' or stocks), but this is not necessarily informative about the amount of private information (Σ_t) in the market.

We may wonder how general these findings are. Specifically, under what conditions the optimal trading strategy of the insider, given knowledge of the current level of noise trading volatility, is independent of the uncertainty about future noise trader volatility. The next example offers a characterization of the equilibrium for more general diffusion dynamics, which helps clarify this further.

3.4. General Diffusion Dynamics

Suppose that volatility follows a strictly positive process of the form:

$$\frac{d\sigma_t}{\sigma_t} = m(t, \sigma^t)dt + \nu(t, \sigma^t)dW_t, \tag{67}$$

where W is a standard Brownian motion independent of Z and m, ν are general processes (that satisfy the technical restrictions of the previous section). The backward equation for $G(t)$ is as before. We can characterize the solution as follows.

Lemma 5. *Suppose that* ν_t *is such that the process* $\xi_t = e^{-\int_0^t \frac{\nu_s^2}{2} ds + \int_0^t \nu_s dW_s}$ *is a martingale*[8], *then a solution to equation (48) is given by the process:*

$$G(t) = \sigma_t^2 A(t)^2, \tag{68}$$

where $A(t)$ *solves the following recursive equation:*

$$A(t)^2 = \tilde{\mathrm{E}}_t \left[\int_t^T e^{\int_t^u 2m_s - \sigma_A(s)^2 ds} du \right], \tag{69}$$

where the expectations is taken with respect to the measure $\tilde{P}$ *equivalent to* P *and defined by the Radon-Nykodim derivative* $\frac{d\tilde{P}}{dP} = \xi_T$, *and where* σ_A *is the diffusion of* $\log A(t)$. *It follows that:*

$$G(t) \leq \mathrm{E}\left[\int_t^T \sigma_u^2 du \right]. \tag{70}$$

Proof 10. *Define* $A(t) = \frac{\sqrt{G(t)}}{\sigma_t}$. *Then plugging into equation (48) we see that* $A(t)$ *solves:*

$$\begin{aligned} A(t) &= \mathrm{E}[\int_t^T \frac{e^{\int_t^u m_s ds}}{2A(u)} \frac{\xi_u}{\xi_t} du] &(71)\\ &= \tilde{\mathrm{E}}[\int_t^T \frac{e^{\int_t^u m_s ds}}{2A(u)} du], &(72) \end{aligned}$$

where we have used, for the first line, the fact that (for $u \geq t$*):*

$$\sigma_u = \sigma_t e^{\int_t^u m_s ds} \frac{\xi_u}{\xi_t}, \tag{73}$$

and, for the second line, Girsanov's theorem. Now, it follows, from the law of iterated

[8] Sufficient conditions, such as the Novikov conditions, are given in LS(2001).

expectation, that $e^{\int_0^t m_s ds} A(t) + \int_0^t \frac{e^{\int_0^u m_s ds}}{2A(u)} du$ *is a* $\tilde{P}$ *martingale, say* $\tilde{M}$*. Thus we have:*

$$dA(t) + m_t A(t) dt + \frac{1}{2A(t)} dt = e^{-\int_0^t m_s ds} d\tilde{M}_t. \tag{74}$$

By Itô's formula we also have $dA(t)^2 = 2A(t)dA(t) + \sigma_A(t)^2 A(t)^2 dt$*, where we define* $\sigma_A(t)$ *to be the diffusion of* $\log A(t)$*. It follows that:*

$$dA(t)^2 = (2m_t - \sigma_A(t)^2) A(t)^2 dt = 2A(t)^2 e^{-\int_0^t m_s ds} d\tilde{M}_t. \tag{75}$$

Integrating, using the fact that $A(T) = 0$*, and assuming sufficient regularity for the stochastic integral to be a martingale, we obtain the result.*

The inequality follows immediately from the fact that $\mathrm{E}[\int_t^T \sigma_u^2 du] = \sigma_t^2 \tilde{\mathrm{E}}[\int_t^T e^{\int_t^u 2m_s ds} du]$.

This result helps us understand when the uncertainty about future noise trading volatility level (ν) will affect the trading strategy of the insider, the price impact measure, and equilibrium prices. In particular, we see that the 'uncertainty irrelevance' result obtained in the previous example, generalizes to rather general diffusion settings, as long as the drift is deterministic:

Corollary 1. *If the expected growth rate of noise trading volatility follows a deterministic process* m_t*, then the process* $G(t)$ *admits the solution:*

$$G(t) = \sigma_t^2 \int_t^T e^{\int_t^u 2m_s ds} du. \tag{76}$$

In turn this implies that private information enters prices at a deterministic rate, and that equilibrium price volatility is deterministic.

It is immediate that, in this (deterministic m) case, the trading strategy of the insider is not affected by uncertainty about future noise trading volatility (as measured by the

ν process). However, as in the previous example, the insider's strategy is a function of the level of noise trading volatility, and thus stochastic.

For the insider to change his strategy depending on the uncertainty about future noise trading volatility, the growth rate of noise trading volatility m_t has to be stochastic. When that is the case, the inequality (70), obtained in the previous lemma, suggests that the agent will act as if there was less cumulative noise trading volatility than a 'myopic' agent who would simply consider the expected remaining total cumulative variance due to noise trading.

To better understand this case, we consider a case where volatility is mean-reverting.

3.5. Mean-reverting noise trading volatility

Consider the case where $x_t = \log \sigma_t$ follows a mean-reverting Ornstein-Uhlenbeck process:

$$dx_t = (-\frac{\nu^2}{2} - \kappa x_t)dt + \nu dW_t. \tag{77}$$

We parametrize the drift of x_t so that, when $\kappa = 0$, volatility is a martingale:

$$\frac{d\sigma_t}{\sigma_t} = -\kappa x_t dt + \nu dW_t. \tag{78}$$

As a result, we can focus on the impact of mean-reversion alone, and use a series expansion in κ around the known solution when $\kappa = 0$ (derived in example 2). The following result characterizes the solution.

Theorem 7. *If the log of noise trading volatility follows a mean reverting process as given in equation (78), then the process $G(t)$ admits the solution:*

$$G(t) = \sigma_t^2 A(T - t, x_t, \kappa)^2, \tag{79}$$

where the function $A(\tau, x, \kappa)$ can be approximated by a series expansion:

$$A(\tau, x, \kappa) = \sqrt{T-t}\left(1 + \sum_{i=1}^{n}(-k\tau)^i \left(\sum_{j=0}^{i} x^j \sum_{k=0}^{i-j} c_{ijk} t^k\right) + O(\kappa^{n+1})\right), \tag{80}$$

where the c_{ijk} are positive constants that depend only on ν^2 and can be solved explicitly.[9] *In that case, private information enters prices at a stochastic rate that depends on the level of noise trading volatility:*

$$\frac{d\Sigma_t}{\Sigma_t} = -\frac{1}{A(T-t, x_t, \kappa)^2} dt. \tag{81}$$

Market depth is stochastic and given by:

$$\lambda_t = \frac{\sqrt{\Sigma_t}}{\sigma_t A(T-t, x_t, \kappa)}. \tag{82}$$

The trading strategy of the insider is:

$$\theta_t = \frac{\sigma_t}{\sqrt{\Sigma_t} A(T-t, x_t, \kappa)}(v - P_t). \tag{83}$$

Stock price dynamics follow a three factor (P, x, Σ) Markov process with stochastic volatility given by:

$$dP_t = \frac{(v - P_t)}{A(T-t, x_t, \kappa)^2} dt + \frac{\sqrt{\Sigma_t}}{A(T-t, x_t, \kappa)} dZ_t. \tag{84}$$

In particular, stock price volatility is a stochastic and tends to be higher when noise trading volatility is higher. The unconditional expected profit at time zero of the insider is $T\sigma_v\sigma_0\frac{A_0}{\sqrt{T}}$.

Proof 11. *To prove this result, we observe that $m_t = -\kappa x_t$ and that x_t has following*

[9] We provide in the appendix the fifth order solution. Higher order expansions can be obtained easily using Mathematica (program available upon request).

dynamics under the $\tilde{P}$ measure:

$$dx_t = (\frac{\nu^2}{2} - \kappa x_t)dt + \nu d\tilde{W}_t, \tag{85}$$

where by Girsanov's theorem we have defined $\tilde{W}_t = W_t - \nu^2 t$ a standard $\tilde{P}$-measure Brownian motion. Thus x_t is a one-factor Markov process under $\tilde{P}$. Using the Markov property for conditional expectations, we guess that the solution to equation (69) is given by a function $A(t, x_t)$.

As shown in the proof of lemma 4, this function satisfies:

$$\tilde{\mathrm{E}}_t \left[dA(t,x_t)/dt + m_t A(t,x_t) + \frac{1}{2A(t,x_t)} \right] = 0. \tag{86}$$

Using Itô's lemma we obtain the following non-linear PDE for $A(T-t,x)$ (where we change variables to $\tau = T - t$ and drop the argument of the function for simplicity):

$$\frac{\nu^2}{2}(A_{xx} + A_x) + -\kappa x(A_x + A) - A_\tau + \frac{1}{2A} = 0 \tag{87}$$

subject to boundary conditions $A(0,x) = 0$. When $\kappa = 0$, the solution is simply $A(\tau, x; \kappa = 0) = \sqrt{\tau}$. Assuming the solution is analytic in its arguments, we seek an series expansion solution of the form given in equation (80) above. Plugging this guess into the left hand side of the PDE and Taylor expanding in κ, we find that each term in the series expansion can be set to zero by an appropriate choice of the constants c_{ijk}. We can thus recursively solve for these constants and obtain an approximate solution to the PDE. In figures 9 in the appendix we plot the 0th, 1st, 2nd and 5th order expansion solution for $\nu = 0.7$ $T = 1$,$\kappa = 0.25$ and for three values of $x_0 = \{-0.3; 0; +0.3\}$.

The first term in the series expansion of the $A(\tau, x, \kappa)$ function is instructive. Indeed, we find:

$$A(\tau, x, \kappa) = \sqrt{\tau}(1 - \frac{\kappa}{2}\tau(\frac{\nu^2\tau}{6} + x)) + O(\kappa^2). \tag{88}$$

This confirms that we need κ to be different from zero for uncertainty about future noise trading volatility to affect the trading strategy of the insider, and equilibrium prices. We see that for a given expected path of noise trading volatility (e.g., setting $x = 0$ where it is expected to stay constant), the higher the mean-reversion strength κ the lower the A function. This implies that mean-reversion tends to lower the profit of the insider for a given expected path of noise trading volatility (compare his profits to the case where $\kappa = 0$).

Further, we see that the function is decreasing in (log) noise-trading volatility if $\kappa > 0$ (we confirm this for higher order expansions). This implies that stock price volatility is stochastic and positively correlated with noise-trading volatility. Equilibrium prices follow a three-factor Markov Bridge process with stochastic volatility. Private information gets incorporated into prices faster the higher the level of noise trading volatility, as the insider trades more aggressively in these states. Note that, since the $A(\tau, \log \sigma, \kappa)$ function is decreasing and convex in volatility, the insider trades more aggressively than in the case where $\kappa = 0$ (where $A(t, \log \sigma)$ is independent of volatility). In these high volatility states, market depth also improves, but less than proportionally to volatility to account for the more aggressive insider trading.

The net effect is that the insider's strategy changes as a function of uncertainty about future noise trading volatility, as the insider can benefit from timing market (liquidity) conditions in this context. In fact, the higher ν^2 the more aggressively does the insider choose to respond a change in noise trading volatility (as A is decreasing in ν^2).

In the last example, we consider a continuous time Markov Chain process for noise-trader volatility. This introduces state-dependent predictability and jumps in noise trader volatility in a simple manner and leads to a tractable illustration of the dynamics of price volatility and private information revelation. We can also simply illustrate the relation between market liquidity states and execution costs to insiders.

3.6. A continuous time Markov chain example

Here we consider a case where M_t is not a continuous martingale and show that we can still derive an equilibrium following the same approach used in the Brownian case above. We assume that uninformed order flow volatility follows a two-state continuous Markov Chain, i.e., there are two regimes $s_t \in [0,1]$ with $\sigma(0) < \sigma(1)$, and the dynamics of the regime indicator are:

$$ds_t = (1 - s_t)dN_0(t) - s_t dN_1(t), \tag{89}$$

where $N_i(t)$ is a standard Poisson counting process with jump intensity η_i respectively.[10]

Then we can show the following. Since the volatility process is Markov, we seek a solution to the BSDE for G of the form $G(t, s_t)$ that satisfies:

$$\sqrt{G(t,s_t)} = \mathrm{E}_t\left[\int_t^T \frac{\sigma(s_u)^2}{2\sqrt{G(u,s_u)}}du\right]. \tag{90}$$

We can characterize it as follows.

Theorem 8. *A solution to (90) is the function* $G(t, s_t) = \mathbf{1}_{\{s_t=0\}} G^0(T-t) + \mathbf{1}_{\{s_t=1\}} G^1(T-t)$*, where the deterministic functions* G^0, G^1 *satisfy the system of ODE given in (91) below, with boundary conditions* $G^0(0) = G^1(0) = 0$.

$$\begin{aligned} G^0_\tau(\tau) &= \sigma(0)^2 + 2\eta_0(\sqrt{G^1(\tau)G^0(\tau)} - G^0(\tau)) \quad (91)\\ G^1_\tau(\tau) &= \sigma(1)^2 + 2\eta_1(\sqrt{G^1(\tau)G^0(\tau)} - G^1(\tau)) \quad (92)\end{aligned}$$

Proof 12. *To prove this results note that if G is a solution to (90) then* $\sqrt{G(t,s_t)} + \int_t^T \frac{\sigma(s_u)^2}{2\sqrt{G(u,s_u)}}du$ *is a martingale, by the law of iterated expectation. Applying Itô's lemma we obtain that G then solves the system of ODE derived above. It is clear that the*

[10]For example, η_0 is the intensity of moving from state 0 to state 1.

boundary condition is $G(T, S) = 0$.

We note that when there is no transition between states $\eta_i = 0$ then the solution reduces to the familiar one obtained in Back (1992), i.e., $G^i(\tau) = \sigma(i)^2(T-t)$. In general, the system of coupled differential equations for $G^i(t)$ $i = 0, 1$ can be easily solved numerically (we have not been able to find a closed-form solution). We note that as maturity approaches, as long as the switching intensities η_1, η_2 are not too (i.e., unboundedly) large, the solution for the price process converges to a pure Brownian bridge as in the continuous time version of the Kyle model presented in Back (1992). However, with more time to go before maturity, the possibility of transitionning from one liquidity state to another changes the optimal strategy of the insider and the price impact function.

For illustration, we choose a period length $T = 1$, $\eta_0 = \eta_1 = 2$ (2 transitions per period), $\sigma(0) = 0.2$ and $\sigma(1) = 0.5$. For these parameter values we report in figure 4 the G-function in the high and low state. As expected, close to maturity the two functions converge smoothly to the lines $\sigma(i)^2(T-t)$ (with $i = 0, 1$) that would prevail, if there were no transitions between states (i.e., the state was absorbing), which also corresponds to the original Kyle model.

Figure 4 shows that, typically, when there is a switch in regime, say from the low to the high volatility regime, the measure of price impact (Kyle's lambda) will jump down, as price impact is lower in the high noise-trading volatility regime. Indeed, recall that $\lambda_t = \sqrt{\frac{\Sigma_t}{G_t}}$ and that since Σ_t is an absolutely continuous process, the immediate effect of an upward jump from G^0 to G^1 is to lower λ. (of course subsequently, in the high noise trading regime, information will be impounded more quickly into prices, leading to a faster drop in Σ_t than in the low volatility regime).

Using the explicit solution for the amount of private information $\Sigma(t) =$

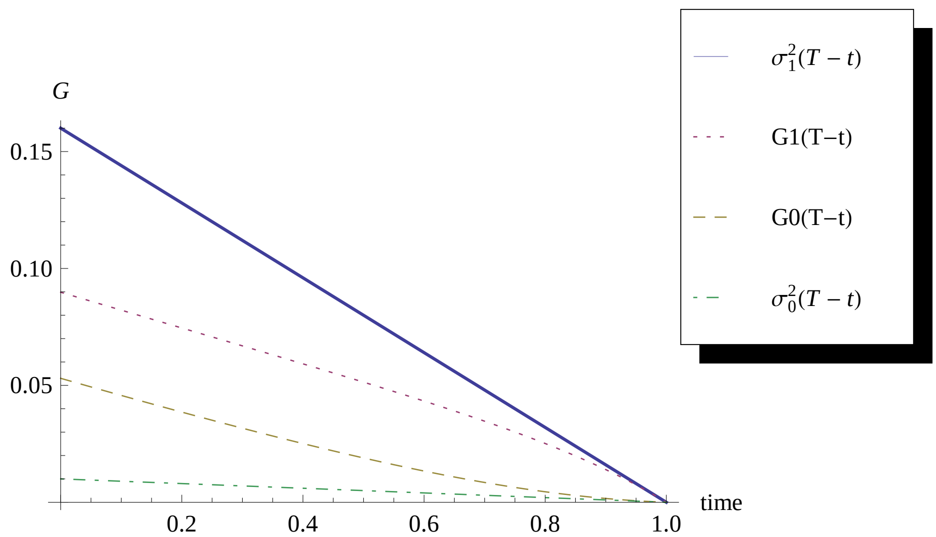

Figure 4: G function in high and low state that solve equation (90). We also plot the lines $\sigma(0)(T-t)$ and $\sigma(1)T-t$, which sandwich respectively $G^0(T-t)$ and $G^1(T-t)$.

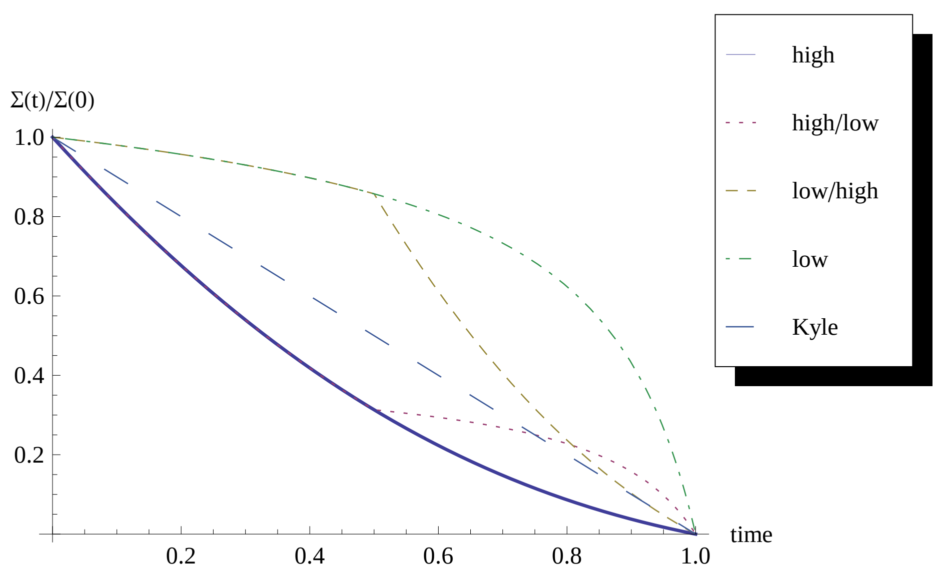

Figure 5: Four paths of the remaining amount of private information $\Sigma(t)/\Sigma(0)$ corresponding to four different noise trader volatility scenarios: (a) start and stay in the high volatility regime until T, (b) start and stay in the low volatility regime until T, (c) start in the high volatility regime and switch to low volatility at $t = 0.5$, and (d) start in low volatility regime and switch to high at $t = 0.5$. We also plot as a benchmark, the Kyle (1985) economy private information decay, which is linear and independent of the noise trader volatility level.

$\Sigma(0)e^{-\int_0^t \frac{\sigma_u^2}{G_u}du}$, we present in figure 5 four paths of $\Sigma(t)$ which depict the revelation of private information in our economy relative to the Kyle (1985) benchmark. We plot $\Sigma(t)/\Sigma(0)$ for the case where noise trading volatility switches to the high regime at date zero and stays there until maturity (high), when it starts in the low regime and stays there until maturity, and when there is a jump at $t = 0.5$ from high to low and low to high respectively. Note that in Kyle, information always decays linearly in time, **irrespective** of the level of noise trader volatility, in the sense that $\Sigma^{kyle}(t)/\Sigma^{kyle}(0) = T - t$. Instead, when noise trading volatility can change stochastically, information flows into prices in a very different fashion. As figure 5 reveals, the posterior variance, when in the high volatility regime, is a decreasing convex function of time, but becomes decreasing concave when there is a switch to the low noise trading regime. The intuition, is that in the low noise trading regime, the insider is playing a waiting game, in the sense that he trades much less aggressively, than he would in the Kyle economy with the same level of volatility. He does so hoping for the high noise trading regime to arrive, where he trades more aggressively, leading to much faster arrival of private information. Of course, if the regime switch does not arrive then ultimately, he will have to become more aggressive so that all his information eventually makes it into prices (see the path marked as 'low' on the graph).

This suggests that all the price impact measures and execution cost measures will be path dependent. For example, we plot in figure 6, for the same four noise trading volatility scenarios, the corresponding path of the price impact ($\lambda(t)$) process. We see that if the economy starts in the high noise trading regime and stays there until maturity, then measured price impact is relatively low and decays steadily (the path is only slightly concave). Instead, if the economy starts in the low noise trading regime, then price impact level is at first only slightly higher than the price impact level in the high noise trading regime, but it increases exponentially as the economy approaches maturity. Similarly, if the regime switches at some point from high to low volatility, then price impact immediately jumps up a little, but subsequently, market depth deteriorates very rapidly as λ increases along a very convex path. This captures intuitively, the submartingale property of λ. On average, execution costs are expected to deteriorate as the economy approaches maturity. Interestingly, note that if the economy is in the high noise trading regime, then measured price impact will be low and decrease steadily at the beginning, even though there is a lot of 'asymmetric information' in the sense that, from figure 5, we see a lot of information getting into prices. Comparing figures 5 and 6 suggests that the level of λ, obtained by 'regressing' stock prices changes on order flow, does not give a valid measure of the amount of private information flowing into prices (as measured by the slope of $\Sigma(t)$).

In figure 7 we plot the volatility of the stock price (which equals $\lambda_t \sigma_t$), for the same four noise trading volatility scenarios. As we see, stock price volatility tends to be higher in the high noise-trading volatility regime. If the economy stays in that regime, then volatility drops steadily. However, if the economy jumps to a low noise trading volatility regime, then stock price volatility jumps down, a large amount, and then subsequently rises rapidly, following an exponential path. These dynamics are intuitive, given the discussion on the dynamics of price impact. In the low noise trading regime, there is less informed trading, thus price responds less to order flow. However, if the higher noise trading volatility regime does not materialize, then, since the same total of amount of private information will make it into price eventually, price impact increases dramatically to reflect the 'catching-up' trading by insiders.

Lastly, from its definition, it is clear that execution costs paid by uninformed noise traders are closely related to the path of stock price volatility (indeed, from their definition, execution costs for insiders are the noise trader volatility weighted integral of

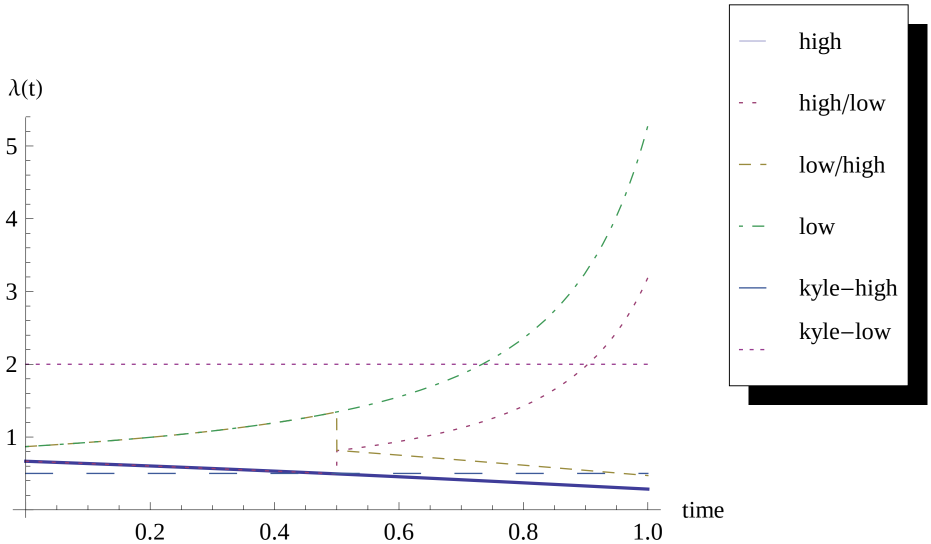

Figure 6: Four separate paths of equilibrium price impact (lambda) dynamics corresponding to (a) start and stay in the high volatility regime until T, (b) start and stay in the low volatility regime until T, (c) start in the high volatility regime and switch to low volatility at $t = 0.5$, and (d) start in low volatility regime and switch to high at $t = 0.5$.

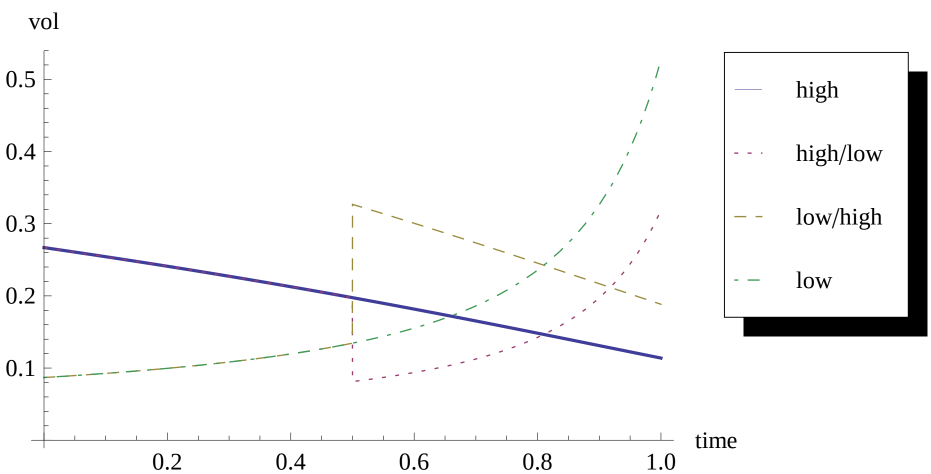

Figure 7: Four separate paths of stock price volatility corresponding to (a) start and stay in the high volatility regime until T, (b) start and stay in the low volatility regime until T, (c) start in the high volatility regime and switch to low volatility at $t = 0.5$, and (d) start in low volatility regime and switch to high at $t = 0.5$.

the area below the path of the stock price volatility). In figure 8 we plot, for the same four scenarios, the path of realized execution costs ($\lambda_t \sigma_t^2$). As pointed out in lemma 3, the total execution costs paid by uninformed at time T is the area below each curve plotted. From the graph it is clear that execution costs are lowest in the low volatility regime scenario, and much higher in the high noise trading volatility regime. We give the corresponding numbers in table 1 below.

This is paradoxical, since as is clear from the table, the high noise trading volatility regime is also the one where the average measured price impact (λ) is lower. Now, one may argue that the difference between the two is simply that there are simply more noise traders in high volatility scenario than in the low volatility scenario, and that therefore it is natural that the total costs are higher. However, if we compare the two other scenarios (high/low to low/high), where arguably, there are the same 'number' of noise traders along each path (in the sense that the cumulative quadratic variation of noise trader order flow is the same across both paths as is confirmed in the third row of table 1), then we see that this is not the only reason. Indeed, execution costs paid by uninformed traders is higher in the low/high than in the high/low scenario (see first row of table 1). Interestingly, we see that the average price impact is higher in the high/low than in the low/high regime, which indicates that focusing on the average level of price impact measures does not capture the realized execution costs. Instead, normalized execution costs, measured as 'volume' weighted price impact normalized by total noise trader volume, is perhaps better informative of the average execution cost for the average uninformed trader. We see in the last row of table 1 that the average price impact thus measured is indeed higher in the low/high than the high/low regime, and indeed, also higher in the low regime than the high regime, indicating the importance of normalizing price impact measures, when there can be variation in uninformed noise trader volatility.

	Noise trading volatility paths			
	high	low	high/low	low/high
Execution costs ($\int_0^T \lambda_t \sigma_t^2 dt$)	0.078	0.017	0.054	0.057
Average price impact ($\int_0^T \lambda_t dt$)	0.487	1.740	1.023	0.853
Total 'number' of uninformed ($\int_0^T \sigma_t^2 dt$)	0.16	0.01	0.085	0.085
Normalized execution costs ($\frac{\int_0^T \lambda_t \sigma_t^2 dt}{\int_0^T \sigma_t^2 dt}$)	0.487	1.740	0.636	0.671

Table 1: This table presents the realized execution costs for uninformed traders depending on various scenarios of realized paths of noise trader volatility. Each path of realized noise trader volatility corresponds to a certain 'number' of uniformed traders arriving to the market. This 'number' is measured by the quadratic variation of the order flow. Normalized execution costs measure the total execution costs divided by the number of uninformed traders.

Unlike in previous literature, execution costs are path-dependent and depend in a complex manner on the realized path of noise trader volatility (and not just on the total cumulative amount of noise trading relative to private information as in Kyle (1985) or Back and Pedersen (1998)).

4. Conclusion

In this paper we have extended Kyle (1985) model of dynamic insider trading to the case where noise trader volatility can change stochastically over time. In equilibrium, we find that the insider adjusts his optimal trading strategy to trade less when noise trading liquidity is lower and more when it is higher. Since market makers anticipate this, in equilibrium, measures of market depth are time varying. Market depth is a martingale and therefore its inverse, price impact, is a submartingale, indicating that on average execution costs are expected to deteriorate over time. If noise trader volatility is predictable, then the equilibrium price exhibits stochastic volatility. It follows a multi-variate 'stochastic bridge' process, which can be seen as a generalization of the classic

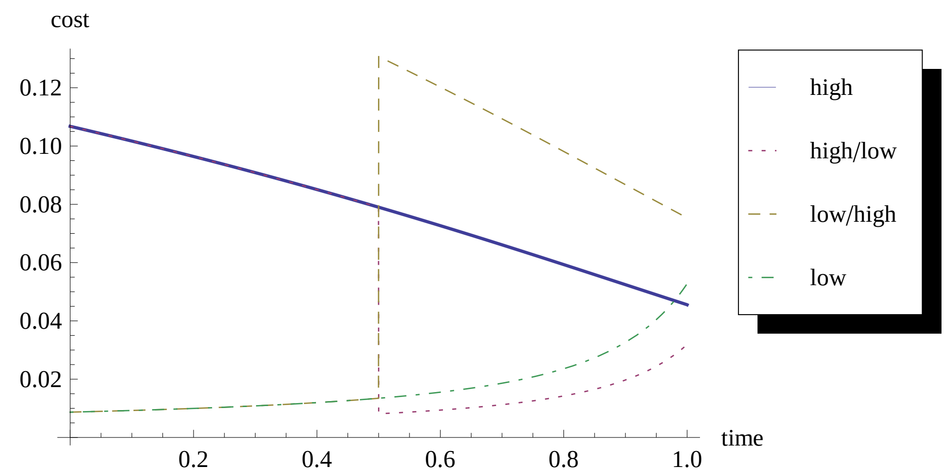

Figure 8: Four separate paths of realized execution costs ($\lambda_t \sigma_t^2$) corresponding to (a) start and stay in the high volatility regime until T, (b) start and stay in the low volatility regime until T, (c) start in the high volatility regime and switch to low volatility at $t = 0.5$, and (d) start in low volatility regime and switch to high at $t = 0.5$. As explained in lemma 3 the area under each path represents the execution costs incurred by uninformed traders.

Brownian bridge to stochastic volatility.

Since the insider trades more aggressively when measured price impact is lower, more information gets into prices when price volatility (and noise trading) is high. Somewhat paradoxically then, when volume is high, measures of execution costs for the uninformed traders (due to 'adverse selection') may be higher, reflecting the fact that insider trades more aggressively, and therefore that more information gets into prices, even though price impact measures, such as Kyle's lambda, might be lower. Some of these results seem consistent with empirical facts documented in the literature (e.g., Collin-Dufresne and Fos (2012) find that informed trade more aggressively when liquidity measures seem better, Madhavan, Richardson and Roomans (1997) find that measured execution costs tend to rise on average over the day, Foster and Viswanathan (1993) find a positive relation between adverse selection measures and volume), but more work remains to be done to test the implications of the model more directly.

5. Appendix

The fifth order expansion of the A function (with $v = \nu^2$).

$$
\begin{aligned}
A(\tau,x,\kappa) = \sqrt{t}\Bigg(1 - \kappa t\left(\frac{\mathrm{v}t}{12}+\frac{x}{2}\right) + \kappa^2 t^2\left(\frac{13\mathrm{v}^2t^2}{1440} + x\left(\frac{\mathrm{v}t}{12}+\frac{1}{6}\right) + \frac{7\mathrm{v}t}{96} + \frac{5x^2}{24}\right) \\
- \kappa^3 t^3\left(\frac{89\mathrm{v}^3t^3}{120960} + x\left(\frac{3\mathrm{v}^2t^2}{320} + \frac{323\mathrm{v}t}{2880} + \frac{1}{24}\right) + \frac{11\mathrm{v}^2t^2}{640} + x^2\left(\frac{59\mathrm{v}t}{1440}+\frac{1}{6}\right) + \frac{3\mathrm{v}t}{80} + \frac{x^3}{16}\right) \\
+ \kappa^4 t^4\left(\frac{1237\mathrm{v}^4t^4}{29030400} + \frac{337\mathrm{v}^3t^3}{161280} + x^2\left(\frac{71\mathrm{v}^2t^2}{16128} + \frac{2593\mathrm{v}t}{34560} + \frac{59}{720}\right) + \frac{6827\mathrm{v}^2t^2}{387072}\right. \\
\left. + x\left(\frac{17\mathrm{v}^3t^3}{24192} + \frac{2657\mathrm{v}^2t^2}{120960} + \frac{737\mathrm{v}t}{8640} + \frac{1}{120}\right) + x^3\left(\frac{\mathrm{v}t}{80}+\frac{59}{720}\right) + \frac{31\mathrm{v}t}{2160} + \frac{79x^4}{5760}\right) \\
- \kappa^5\left(t^5\right)\left(\frac{6299\mathrm{v}^5t^5}{3832012800} + \frac{193\mathrm{v}^4t^4}{1244160} + \frac{51709\mathrm{v}^3t^3}{16588800} + x^3\left(\frac{601\mathrm{v}^2t^2}{483840} + \frac{4673\mathrm{v}t}{161280} + \frac{59}{960}\right) + \frac{18703\mathrm{v}^2t^2}{1451520} + \right. \\
x^2\left(\frac{4241\mathrm{v}^3t^3}{14515200} + \frac{7129\mathrm{v}^2t^2}{580608} + \frac{9127\mathrm{v}t}{120960} + \frac{11}{360}\right) + x\left(\frac{287\mathrm{v}^4t^4}{8294400} + \frac{49439\mathrm{v}^3t^3}{21772800} + \frac{319777\mathrm{v}^2t^2}{11612160} + \frac{2293\mathrm{v}t}{48384} + \frac{1}{720}\right) \\
\left. + x^4\left(\frac{431\mathrm{v}t}{161280} + \frac{1}{40}\right) + \frac{\mathrm{v}t}{224} + \frac{3x^5}{1280}\right)\Bigg) + O(\kappa^6)
\end{aligned}
$$

We illustrate the convergence of the expansion in the following figures.

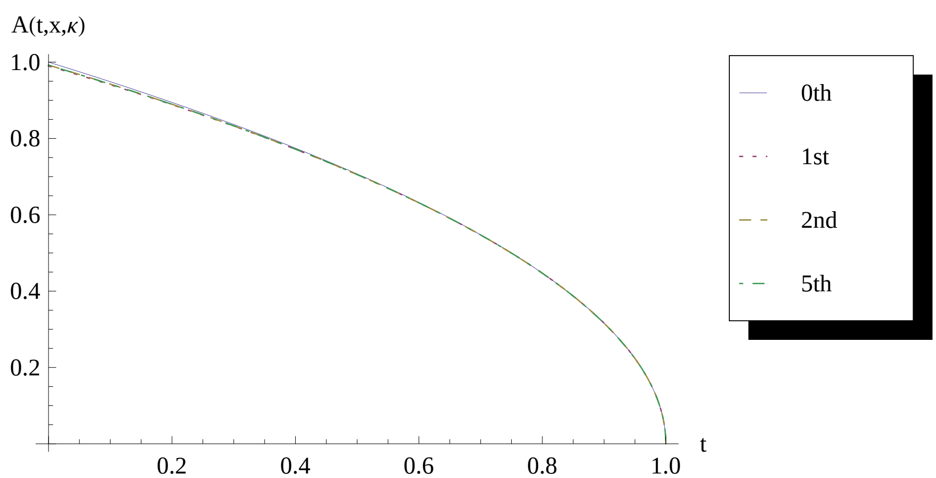

Figure 9: A function expansion solution given in equation (80) for different order (0,1,2,5) of the expansion for $x_0 = 0$. Other parameter values are $\kappa = 0.25, \nu = 0.7, T = 1$.

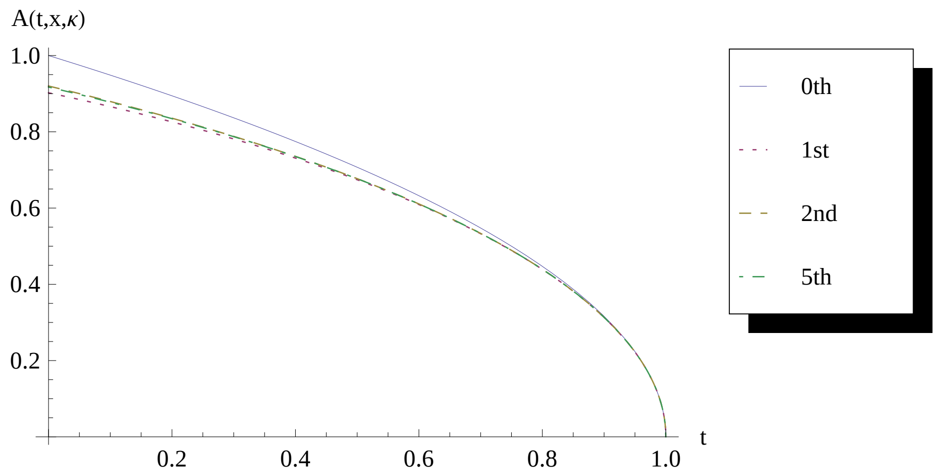

Figure 10: A function expansion solution given in equation (80) for different order (0,1,2,5) of the expansion for $x_0 = +0.7$. Other parameter values are $\kappa = 0.25, \nu = 0.7, T = 1$.

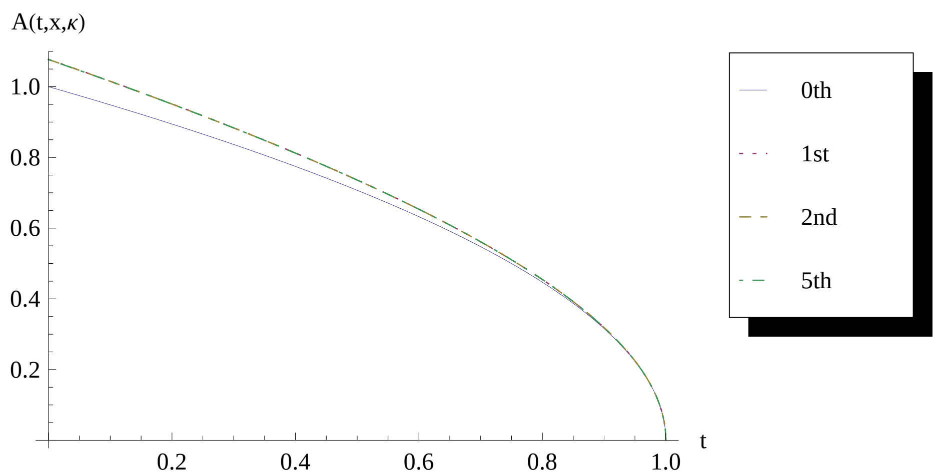

Figure 11: A function expansion solution given in equation (80) for different order (0,1,2,5) of the expansion for $x_0 = -0.7$. Other parameter values are $\kappa = 0.25, \nu = 0.7, T = 1$.

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