Heterogeneous Beliefs about Rare Event Risk in the Lucas Orchard*

Ilaria Piatti[†]

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Abstract

This paper investigates the asset pricing implications of investor disagreement about the likelihood of a systemic disaster. I specify a general equilibrium Lucas endowment economy with multiple trees and heterogeneous beliefs about the risk of rare systemic events; the aim is to understand how risk-sharing mechanisms and fear affect equity and variance risk premia, at an aggregate level and in the cross section of stock returns. I first identify a state-dependent conditional link between equity and variance premia, that changes with the cross-sectional distribution of agent consumption. Second, although the risk-neutral stock returns correlation is increasing in the share of pessimistic agents, the correlation computed under the objective measure is not; the result is a countercyclical correlation risk premium. Third, as the number of assets increases, the aggregate variance premium is driven almost entirely by fear of systemic disasters. Empirically, I find that, as anticipated by the model, the variance premium's power to predict future excess returns is greater during times of financial distress, which typically feature more disagreement among investors. This result holds especially for small stocks, which are more sensitive to systemic rare event risk.

Keywords: heterogeneous beliefs, systemic disasters, Lucas orchard, variance risk premium, correlation risk premium, predictability

1 Introduction

Since the outbreak of the global financial crisis in 2008, *tail* or *disaster* risk—understood as the potential presence of infrequent adverse events of extreme magnitude—has been a concern for academics and investors alike. For instance, Hoang Le Huy, head of fixed income and event strategies at Schroders New-Finance Capital, a London-based fund of funds states:[1] "You need to hedge against disaster scenarios. Black swan events are at the forefront for a lot of investors right now. It is not something that people take lightly. A lot of tail risk funds were built on the back of the 2008 disaster." Numerous studies show that even a small probability of an extreme event in economic fundamentals can have significant effects on asset prices. These extreme events are rare by definition and so accurately estimating their likelihood is difficult, which is a natural source of investor disagreement over perceived tail risk. Such heterogeneity of beliefs about disasters suggests that belief-driven risk sharing could explain the pricing of rare event risk. Compensation for disaster risk contributes to a significant fraction of expected returns on equity and pure variance positions. Thus, a better understanding of disaster risk premia should help explain the dynamics of equity and variance risk premia and the nature of their comovement. This paper studies, both theoretically and empirically, how agent disagreement about disaster risk affects excess return dynamics and the relation between the equity and the variance risk premia both for the market portfolio and the cross section of stocks.

I develop a general equilibrium Lucas (1978) economy with multiple assets and heterogeneous beliefs in which the premia for equity and variance positions and their comovement are endogenously driven by investor disagreement and the cross-sectional distribution of consumption. The model suggests a stronger predictive power of variance risk premium for future excess returns in periods during which pessimists have a relatively large consumption share—that is, in bad states of the economy, which are also characterized by higher (absolute) values of the variance risk premium. Accordingly, I find empirically that variance risk premia and their predictive power for future excess returns are concentrated in phases of substantial disagreement among investors. In these phases, regression coefficients and R^2 are particularly large for small stocks, whose returns are more dependent on the compensation for systemic rare event risk. Therefore, exposure to aggregate variance risk could partially explain the *size effect* (i.e. the observation that smaller firms have higher returns on average), which actually seems to be most pronounced during periods in which pessimists hold a large fraction of the aggregate endowment. The time variation in the sign and strength of the predictive power of the aggregate variance premium—both for future excess market returns and for individual stocks—is a challenge for existing consumption-based asset pricing models. The model I posit addresses these empirical challenges, and I provide a structural explanation based on the role of risk sharing between agents who disagree.

The main ingredients of the model are the following. First, I consider an endowment economy with a single consumption good but multiple trees.[2] The endowment processes follow a geometric Brownian

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[1] See Risk.net.

[2] In this context, *trees* are assets and a collection of trees is an *orchard*. See e.g. Martin (2013).

motion with the addition of idiosyncratic and systemic jump components. The presence of multiple trees allows me to study the relation between equity premia in the cross section and the aggregate variance premium, together with the determinants of comovement between trees. Second, two groups of investors, who have constant relative risk aversion (CRRA) preferences over consumption, have different beliefs about the likelihood of a rare systemic event. Disagreement is an important source of non continuous variation in the variance risk premium dynamics. The observed market variance premium is, in fact, highly time varying; periods of small and smooth premium alternate with periods in which the variance premium is larger (in absolute value) and more volatile. The presence of disagreement about the intensity of disasters also allows the variance risk premium to switch sign in certain phases, mainly for small individual stocks; such switching is consistent with the empirical evidence. Third, I assume that the intensity of the systemic jump process is time varying and proportional to an exogenous state variable that can be interpreted as a continuous signal reflecting the state of the economy. The two agents disagree on the coefficient of proportionality, so that the absolute difference in perceived expected growth rates is also proportional to the exogenous state variable and never switches sign. This simple specification of disagreement can be considered as a reduced-form way to capture several empirical regularities of differences in opinion which have been recently documented.

Borrowing from the solution methods proposed for the Lucas Orchard by Martin (2013) and from methods used in the single-asset difference in beliefs model of Chen, Joslin, and Tran (2012), I derive semi-closed-form expressions for the stock prices in my multiple trees economy with heterogeneous beliefs. Price-dividend ratios of individual stocks and parameters in the price process dynamics depend on the consumption share of the two agents, on the state variable driving time-varying intensities, and on the dividend share distribution.

Using the model solution, I derive a number of testable predictions. First, the equity (variance) risk premium of an individual stock tends to increase (decrease) with its dividend share and with the consumption share of the pessimistic agent; these phenomena can be explained by the risk-sharing behavior of disagreeing investors. Moreover, as noted above, the variance risk premium can switch sign— in particular for small stocks—when optimists consume a large fraction of the aggregate endowment and disagreement is large enough. In line with the data, the variance premium is time varying; it alternates phases of small and smooth premia with periods in which the variance premium is larger (in absolute value) and more volatile, where the change in regime is driven by an abrupt change in the cross-sectional distribution of agent consumption. Second, the model-implied correlation risk premium inherits these features because, consistently with the empirical findings of Driessen, Maenhout, and Vilkov (2012), the index variance risk premium is largely due to a covariance premium, mainly when assets are relatively evenly distributed or the number of stocks in the economy is large. While the cross-sectional distribution of agent consumption mainly affects the risk-neutral stock return correlation, the physical correlation is relatively insensitive to it, which leads to a countercyclical correlation risk premium. Third, rare event risk implies a tight link between the equity and the variance risk premia, both for the market and for the

cross section of stock returns. This link provides the basic intuition for the role of the variance premium in predicting future excess returns. However, standard predictive regressions imply an unconditionally linear relation between equity and variance risk premia, whereas in the model the regression coefficients are stochastic and depend on the asset's dividend share and the agents' consumption share. I show by simulation that the aggregate variance premium's power to predict future excess returns is stronger when the consumption share of the pessimist is larger, i.e., in bad states of the economy. At a disaggregate level, the predictive power of the variance risk premium is especially great for small stocks. Fourth, I consider the special case of a large diversified economy in which the number of stocks approaches infinity and all stocks have the same dividend share, which approaches zero. In this case, only systemic risk is priced and the relation between equity and variance risk premia is conditionally linear. Moreover, infinitely small assets still earn a risk premium owing to the presence of systemic rare event risk.

The model's main predictions are tested using the aggregate $S\&P500$ composite index as a proxy for the aggregate market and the return time series of all its single constituents, as well as CRSP cap-based portfolio returns, to analyze cross-sectional implications and the differential effects of small versus big stocks, based on monthly data from January 1990 through December 2011. The test results confirm that the index variance premium's ability to predict future excess returns is (a) time varying for the market and for single stocks or stock portfolios and (b) stronger during periods of financial distress. Such periods are characterized by large (absolute) variance premia and substantial investor disagreement, which is proxied by the dispersion in one-year-ahead forecasts of real GDP growth from the BlueChip Economic Indicator. The predictive power of the variance premium is stronger (on average) for small stocks, which have returns that depend more on the compensation for systemic rare event risk. For example, the adjusted R^2 of a standard predictive regression of excess six-month returns on the aggregate variance premium is about 63% larger for the small-cap portfolio than for big caps. The difference between small- and big-cap portfolios is particularly evident in periods of high disagreement. Intuitively, investors will require higher return from assets that are more sensitive to systemic disaster risk.[3] However, this reasoning holds only when the perceived premium for systemic jumps is sufficiently large. The model suggests that the systemic jump premium component can even have a negative effect on a stock's excess returns if the pessimists' consumption share is low enough. Thus the size premium could move in opposite directions depending on what agent type dominates the market. This finding is consistent with the mixed results reported in the empirical literature on size premia.

The rest of the paper is organized as follows. Section 2 provides a literature review. Section 3 introduces the basic model setup as well as the optimal consumption allocation, market prices of risk, and equilibrium market prices. Section 4 analyzes the properties of the equity and variance risk premia, of their relationship, and the correlation risk premium. It also studies the case of an infinitely large and diversified economy. Section 5 describes the data and tests empirically the main implications of the

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[3]The empirical literature on the size premium identifies several reasons why small stocks are more sensitive to systemic risk. One possible explanation is that small firms are more affected by tight credit conditions.

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model. Section 6 concludes and discusses possible model extensions and directions for future research. All proofs can be found in the Appendix.

2   Literature Review

This paper is related to several different strands of the literature. The first is the growing research on asset pricing with multiple trees. Cochrane, Longstaff, and Santa-Clara (2008) highlight the asset pricing implications of a two-trees Lucas (1978) economy with a log-utility representative investor. Martin (2013) introduces multiple Lucas trees (Lucas orchard) following jump-diffusion processes and a representative agent with power utility. Chabakauri (2013) considers two trees and two CRRA investors with heterogeneous risk aversions and portfolio constraints, and he looks at the effects on return correlations and volatilities. Buraschi, Trojani, and Vedolin (2013) specify a diffusive two-trees model with heterogeneity in beliefs; they characterize the relation between the difference in opinions, volatility and correlation risk premia of index and individual options. In contrast to previous papers, I specify a collection of Lucas trees with rare disasters and heterogeneous beliefs about the intensity of systemic rare events, with the goal of studying the implications for equilibrium risk premia and for the relation between the market variance risk premia and excess returns. Multiple trees allow me to analyze the contribution of a premium for covariance risk to this predictive relation. This insight is motivated by the empirical evidence in Driessen, Maenhout, and Vilkov (2012) that the index variance risk premium is largely due to the high price of correlation risk and that option-implied correlations have remarkable predictive power for future stock market returns. In my model, covariance risk can contribute to a large fraction of the aggregate variance premium when the economy is dominated by pessimistic agents. In such states, fear of systemic disasters requires large compensation for both equity and variance risks, leading to a strong comovement between equity and variance risk premia.

The predictive relation between the market variance risk premium and excess returns was first observed by Bollerslev, Tauchen, and Zhou (2009), who provide empirical evidence that the variance risk premium accounts for a nontrivial fraction of the time-series variation in post-1990 aggregate stock market returns at short horizons. They motivate this link theoretically in a long-run risk model with stochastic volatility of consumption growth volatility. Londono (2011) extends that model to an international setting and provides evidence that the US variance premium predicts local and foreign equity returns. Consistently with the implications of a consumption-based model along the lines of Bollerslev, Tauchen, and Zhou (2009), Bali and Zhou (2011) find that the variance risk premium explains both the time-series and cross-sectional variation in stock returns. In such models, recursive preferences are crucial to generate a premium for stochastic volatility of consumption growth volatility, which then drives both the equity and variance risk premia. Yet, Wu (2012) observes that there is no empirical correlation between the variance risk premium and the volatility of consumption growth volatility. Moreover, Boller-

slev and Todorov (2011) find that more than half of the variance risk premium is driven by disaster risk and suggest that equilibrium-based asset pricing models should accommodate large and time-varying compensation for rare disasters. A step in this direction is taken by Drechsler (2011) and Drechsler and Yaron (2011), who incorporate time-varying uncertainty into a long-run risk model—with jumps in expected growth and growth volatility processes—and are able to fit sample moments of the variance risk premium.[4] Intuitively, explaining the existence and properties of variance risk premia requires that equilibrium models endogenously generate time-varying non normality in returns. Following this idea, Bekaert and Engstrom (2010) propose a consumption-based model with preferences as in Campbell and Cochrane (1999) but with nonlinear consumption growth. This model posits two types of shocks (good and bad) that are drawn from potentially skewed and fat-tailed distributions and have a time-varying relative importance. However, none of the cited attempts to explain variance risk premia in a general equilibrium setting truly accounts for rare disasters, defined as jumps in realized (not expected) endowment growth.

The idea that the possibility of sudden downward jumps in the endowment may help explain the equity premium puzzle dates back to Rietz (1988). More recently, Gabaix (2012) and Wachter (2013) resolve several asset pricing puzzles by including a time-varying risk of disasters in otherwise standard models. The literature on rare disasters does not seek to explain the variance risk premium's puzzling dynamics or its ability to predict future excess returns.[5] This paper seeks to fill that gap starting from a general equilibrium model in which two sets of agents have different beliefs about the probability of a disaster occurring.

Previous papers have studied the disagreement that surrounds assessments of disaster risk. Dieckmann (2011) provides an equilibrium model in which log-utility investors have heterogeneous beliefs about the likelihood of rare events; he explores the asset pricing implications of this setup in an incomplete capital market as well as the effects of market completion. Chen, Joslin, and Tran (2012) consider a complete market setting and assume that two CRRA agents disagree about rare event risk. They show that the relation between the disaster risk premium and the extent of disagreement about disaster risk is highly nonlinear; a small proportion of optimistic investors can greatly attenuate the impact of disaster risk on stock prices. I contribute to this literature along several dimensions. First, I study the effects of disagreement on variance risk premia and its predictive power for excess returns. Second, using the multiple trees setting I study the cross-sectional implications of heterogeneous rare event risk. Third, I test empirically the model's main predictions. Finally, I use a specification of disagreement that is consistent with several empirical regularities of differences in opinion. Patton and Timmermann

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[4]Other recent papers using long-run risk models to explain variance risk premia include Zhou (2010) and Zhou and Zhu (2010). Jin (2013) compares several calibrated specifications of long-run risk models, with and without jumps in log-run expected consumption growth and consumption volatility, and argues that jumps in volatility are crucial for explaining variance risk premia and for generating a degree of return predictability that is consistent with the data.

[5]A recent exception is Kim (2013), who uses multiple regimes to model an endowment economy with time-varying likelihood of disasters.

(2010) and Buraschi, Trojani, and Vedolin (2011) show that differences in beliefs are highly time varying and countercyclical. Moreover, Patton and Timmermann (2010) suggest that there is a strong negative correlation between dispersion and consensus forecast on GDP growth. They also find that forecasters' view are persistent—in other words, they tend to be consistently optimistic or pessimistic. In my model, disagreement is countercyclical whereas the average belief about expected consumption growth is pro-cyclical; these dynamics lead to a perfect negative correlation between consensus and dispersion, whose persistence is guaranteed by a positive exogenous state variable.

The relation between the equity and variance risk premium reflects properties of the equilibrium risk–return trade-off. Instead of taking the physical expectation and variance of returns, the volatility risk premium considers the difference between physical and risk-neutral expectation of future volatility, thus isolating the priced component of volatility risk. Hence this paper relates also to the large literature on a time-varying risk–return trade-off. For example, Brandt and Wang (2010) estimate the monthly market risk–return relationship from the cross section of equity returns and show that this relationship is usually positive but varies considerably over time. They also show that the coefficient relating the market risk premium to the conditional market volatility exhibits a countercyclical pattern. Interestingly, Yu and Yuan (2011) find a positive risk–return trade-off when sentiment is low, but no relation when it is high.[6] These authors argue that this result is a challenge for traditional asset pricing theories. Similarly, I find that the relation between equity and variance risk premium is evident mainly when pessimists account for a large fraction of aggregate consumption. Hong and Sraer (2012) also study the link between the risk–return relationship and divergence of opinion in the cross section; they show that the risk–return relationship for single stocks can flip from positive to negative when investor disagreement over the asset value is large enough. Anderson, Ghysels, and Juergens (2009) augment the typical risk–return trade-off model with a measure of uncertainty that is based on the level of disagreement among professional forecasters. They find stronger empirical evidence for an uncertainty–return trade-off than for the traditional risk–return trade-off.

Finally, my results on the connection between the cross section of excess stock returns and the aggregate variance premium are related to the literature on the size effect. Lemmon and Portniaguina (2006) demonstrate a negative relation between the size premium and consumer confidence. In fact, the size effect (whereby smaller firms have higher returns on average) seems to be concentrated in periods characterized by large disagreement. Intuitively, investors tend to require higher returns from assets that are more sensitive to systemic disaster risk. This intuition is well explained by Jagannathan and Wang (1996), who argue that the market beta of firms with a greater likelihood of financial distress (e.g., small firms) is more sensitive to changes in the business cycle. Investor sentiment is thus related to time variation in the expected returns of those firms because such sentiment forecasts future business conditions. However, this reasoning holds only when the perceived systemic jump premium is high. My model indicates that the jump premium component can actually have a negative effect on the stock's

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[6]Yu and Yuan (2011) use the sentiment index by Baker and Wurgler (2006) to identify low and high sentiment periods.

excess returns if the consumption share of the pessimists is sufficiently low. Therefore, the size premium can move in opposite directions, depending on which agent type dominates the market, consistently with the mixed results of the later empirical research on the size effect.[7]

3 An Economy with Multiple Trees and Heterogeneous Beliefs about Systemic Disasters

This section introduces the model, which is a simple, continuous-time generalization of the standard Lucas (1978) endowment economy. The model incorporates rare disasters and heterogeneous beliefs about the probability of a common jump in N Lucas trees. For notational convenience, vectors and matrices are denoted by bold symbols.

Two agents $(i = A, B)$ observe the dividend stream produced by each tree, D_j, with the following exogenous dynamics:

$$\frac{dD_j(t)}{D_j(t)} = \mu_j dt + \sigma_j dW_{jt} + k_j dN_{jt} + k_j dN_{ct}, \quad j = 1, \ldots, N, \tag{1}$$

where $\mathbf{W}_t = (W_{1t}, W_{2t}, \ldots, W_{Nt})'$ is an N-dimensional standard Brownian motion driving regular economic risk, rare event risk enters through the Poisson processes $\mathbf{N}_t = (N_{1t}, N_{2t}, \ldots, N_{Nt})'$ and N_{ct} with respective intensities $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)'$ and $\lambda_c(t)$. Thus each stock has both an idiosyncratic and a systemic event risk component. Namely, the jump in the dividend growth of stock j is idiosyncratic if driven by a jump in N_{jt} whereas jumps in N_{ct} are common to all stocks. For simplicity, and since the goal is to understand the asset pricing implications of heterogeneous beliefs on the probability of a common jump, I assume that the intensity of the systemic Poisson process is time varying while all other parameters, including the idiosyncratic jump intensities, are constant.[8] Furthermore, the coefficients μ_j, σ_j, k_j, and λ_j—which represent, respectively, the expected growth rate and volatility of dividend growth without jumps, the jump size, and the idiosyncratic jump intensity—are assumed to be identical for all trees in the economy (hence I will suppress their subscript j). The jump size k is restricted to be negative and strictly less than 1 in absolute value; this ensures that dividend processes are positive.[9]

To focus on the effects of heterogeneous systemic rare event risk on risk premia, I assume that agent beliefs differ only with respect to the systemic rare event intensity $\lambda_c(t)$, which is a function of an exogenous affine state variable $X(t)$. In particular, agent i believes that the common jump frequency is

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[7]See e.g. Crain (2011) for a review of the size effect.

[8]Wachter (2013), among others, underlies the importance of taking into account time variation in the probability of rare disasters to help explain, e.g., time variation in the equity premium and the excess volatility puzzle, while Berkman, Jacobsen, and Lee (2011) provide empirical support for time-varying rare disaster intensity.

[9]The assumption of constant jump size could be relaxed, but it helps to maintain tractability and to isolate the effect of disagreement about rare event intensity. In single-asset models, Wachter (2013) assumes a lognormal distribution for the jump size, Drechsler and Yaron (2011) and Jin (2013) use Gamma distributions, and Gabaix (2012) and Tsai and Wachter (2013) assume a power law distribution—in line with Bollerslev and Todorov (2011)'s nonparametric evidence that the tails of the risk-neutral distribution of returns decay according to a power law.

given by[10]

$$\lambda_c^i(t) = \beta^i X(t), \tag{2}$$

for $i = A, B$, where $X(t)$ follows a CIR process,

$$dX(t) = \varphi[1 - X(t)]dt + \sigma_X \sqrt{X(t)}dW_t^X, \tag{3}$$

for W_t^X a standard Brownian motion that is independent of $\mathbf{W}_t$. This assumption ensures positivity of the intensity of a common jump under each agent's beliefs, which also follows a CIR process.[11] I assume that the long-term mean of X is equal to 1, so that β^i represents the expected systemic rare event intensity perceived by agent i.

The probability measures of the two agents are equivalent because they agree on null sets. Hence, their Radon–Nikodym derivative $\phi(t) = d\mathbb{P}^B/d\mathbb{P}^A$ exists and has been shown by Chen, Joslin, and Tran (2010) to have the following dynamics:

$$\frac{d\phi(t)}{\phi(t)} = (\beta^A - \beta^B)X(t)dt + \left[\frac{\beta^B}{\beta^A} - 1\right] dN_{ct}. \tag{4}$$

If the agents observe a common jump (i.e. if $dN_{ct} = 1$) then the likelihood ratio jumps by a factor of β^B/β^A. If agent B is optimistic—which means he believes that the probability of a systemic (negative) jump is lower (i.e., $\beta^B < \beta^A$, as I assume throughout the paper)—then ϕ's jump in response to systemic disaster is a downward one. That being said, the absence of systemic jumps over a period of time is more consistent with the optimist's beliefs and so the likelihood ratio increases deterministically at a rate $\lambda_c^A - \lambda_c^B$. Note that, even in the case of constant systemic rare event intensity (i.e., X constant) the state variable ϕ varies over time and decreases dramatically following a disaster.

3.1 Dividend shares and consumption dynamics

I consider an endowment economy in which all trees produce the same perishable consumption good; therefore, aggregate consumption equals the sum of all dividends:

$$C(t) = \sum_{j=1}^{N} D_j(t). \tag{5}$$

Let s_j be the share of consumption contributed by stock j,

$$s_j = \frac{D_j(t)}{C(t)}. \tag{6}$$

An application of Itô's lemma to Equation (1) gives its dynamics:

$$ds_j = \sigma^2 s_j \left(\sum_{i=1}^{N} s_i^2 - s_j \right) dt + \sigma s_j \left(dW_{jt} - \sum_{i=1}^{N} s_i dW_{it} \right) + s_j \frac{k(1 - s_j)}{ks_j + 1} dN_{jt} - s_j \sum_{i \neq j} \frac{ks_i}{ks_i + 1} dN_{it}. \tag{7}$$

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[10]Benzoni, Collin-Dufresne, Goldstein, and Helwege (2012) assume a similar dynamics for a country's default intensity and use a β parameter that depends on the state of the world. They then employ learning to capture contagion effects in the perceived default intensities of different countries.

[11]Wachter (2013) and Chen, Joslin, and Tran (2012) also assume CIR processes for the rare event intensity. Chen, Joslin, and Tran (2012) include disagreement directly in the long-run average jump intensity whereas here the proportionality coefficient β^i is used to introduce disagreement.

Intuitively, the dividend share of asset j increases when there is a positive Brownian shock to its dividend growth dynamics or an idiosyncratic disaster involving any of the other dividend processes; the share decreases in response to an idiosyncratic jump in its own dividend growth. Systemic jumps do not affect dividend share dynamics because such jumps are assumed to have the same impact on all dividend processes. The drift in Equation (7) is zero when $s_j = 0$, $1/N$, or 1, and the dividend share distribution is not stationary because one asset ultimately becomes dominant in the market; that is, $ds_j = 0$ for $s_j = 0, 1$.[12]

By construction, the dividend shares s_j sum to 1 and the $N-1$ state variables s_j for $j = 2, \ldots, N$ are enough to describe the relative size of the N trees. Asset prices will depend on these $N-1$ dividend shares, but Martin (2013) argues that it is often more convenient to use a monotonic transformation of these state variables,

$$u_j = \ln \frac{s_j}{s_1}, \tag{8}$$

which measures the size of asset j relative to asset 1. As s_j ranges from 0 to 1, u_j can take all values on the real line. Applying Itô's lemma to the definition in Equation (8) while assuming symmetric assets, we obtain the dynamics

$$du_j = d \ln D_j - d \ln D_1 = \sigma(dW_{jt} - dW_{1t}) + \ln(k+1)(dN_{jt} - dN_{1t}) \tag{9}$$

for $j = 2, \ldots, N$. In matrix notation, the dynamics of $\mathbf{u} = (u_2, \ldots, u_N)'$ is given by

$$d\mathbf{u} = \sigma \mathbf{U} d\mathbf{W}_t + \ln(k+1)\mathbf{U} d\mathbf{N}_t, \tag{10}$$

where $\mathbf{U}$ is a $(N-1) \times N$ matrix:

$$\mathbf{U} \equiv \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

From Equation (5), the dynamics of aggregate consumption growth is given by

$$\frac{dC(t)}{C(t)} = \mu dt + \sigma \sum_{j=1}^{N} s_j dW_{jt} + k \sum_{j=1}^{N} s_j dN_{jt} + k dN_{ct}. \tag{11}$$

Observe that even if agents agree on μ, the growth rate of consumption in *normal* times, disagreement about the systemic rare event intensity leads to disagreement about the total expected growth rate,

$$\mu_C^i = E_t^i \left[\frac{dC(t)}{C(t)} \right] = \mu + k(\lambda + \lambda_c^i(t)). \tag{12}$$

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[12]This feature is shared by many of the literature's general equilibrium models that involve two or more trees. See for example Cochrane, Longstaff, and Santa-Clara (2008), who discuss properties of the dividend share dynamics in the case of two trees.

Here $E_t^i(.)$ denotes conditional expectation under the probability measure $\mathbb{P}^i$, which summarizes agent i's beliefs. Thus, the difference in expected growth rates can be expressed in terms of the dispersion in beliefs,

$$\mu_C^B - \mu_C^A = k(\lambda_c^B(t) - \lambda_c^A(t)) = -k(\beta^A - \beta^B)X(t), \tag{13}$$

which is linear in the exogenous state variable X.

This simple specification of disagreement is a parsimonious way to capture several empirical regularities of differences in opinion that have been reported recently. Patton and Timmermann (2010) and Buraschi, Trojani, and Vedolin (2011) show that differences in beliefs are highly time varying and countercyclical. Moreover, Patton and Timmermann (2010) suggest that (a) there is a strong negative correlation between belief dispersion and a consensus forecast of GDP growth and (b) forecasters' views tend to be consistently optimistic or consistently pessimistic. In the model, disagreement is countercyclical if the state variable X is interpreted as an exogenous continuous signal about the state of the economy. The average belief as regards expected consumption growth is a decreasing function of X, while the absolute difference in perceived expected growth rates is increasing in X; the result is a perfect negative correlation between consensus forecast and the dispersion in forecasters' beliefs, the persistence of which is guaranteed by the positivity of X.[13]

3.2 Agent optimization problem

Agents have a constant relative risk aversion (CRRA) utility over consumption with finite horizon T:

$$U^i(C^i(t)) = \frac{C^i(t)^{1-\gamma}}{1-\gamma} \tag{14}$$

for $i = A, B$; here γ is the coefficient of relative risk aversion, which is assumed to be identical across agents.[14] If we assume complete markets and use martingale techniques (see e.g. Cox and Huang (1989)) then agent i's optimization problem can be written in static form as

$$J^i = \max_{C^i} E^i \left[\int_0^T e^{-\delta t} U^i(C^i(t))\ dt \right], \quad \text{s.t.} \quad E^i \left[\int_0^T \eta^i(t)C^i(t)\ dt \right] \le W^i(0). \tag{15}$$

Here δ is the time preference rate and $\eta^i(t)$ is the state price density of agent i, whose dynamics is given by

$$\frac{d\eta^i(t)}{\eta^i(t)} = -r(t)dt + \left[\sum_{j=1}^N (\lambda - \lambda_j^Q(t)) + (\lambda_c^i(t) - \lambda_c^Q(t)) \right] dt - \boldsymbol{\theta}(t)' d\mathbf{W}_t + \sum_{j=1}^N \left(\frac{\lambda_j^Q(t)}{\lambda} - 1 \right) dN_{jt} + \left(\frac{\lambda_c^Q(t)}{\lambda_c^i(t)} - 1 \right) dN_{ct}. \tag{16}$$

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[13]Disagreement could instead switch sign in the single-asset belief disagreement model of Chen, Joslin, and Tran (2012), even if disaster intensities follow CIR processes, because these authors introduce disagreement directly in the long-run average jump intensity.

[14]Dieckmann and Gallmeyer (2005) introduce heterogeneity only through different levels of relative risk aversion and study equilibrium allocations. Chabakauri (2013) considers two trees and two CRRA investors with heterogeneous risk aversions and portfolio constraints, and he examines the effects on return correlations and volatilities. Chen, Joslin, and Tran (2012) argue that combining heterogeneous beliefs about disasters and different risk aversions can amplify the effects of risk sharing but does not qualitatively change basic asset pricing results.

In this expression, the N-vector $\boldsymbol{\theta}$ is the market price of regular economic risk associated with Brownian motion $\mathbf{W}(t)$, while λ_j^Q and λ_c^Q are the risk-neutral rare event intensities associated with the respective Poisson processes $N_j(t)$ and $N_c(t)$.[15] Agents are assumed to be initially endowed with a fraction x_s^i of each stock; that is, $W^i(0) = x_s^i \sum_{j=1}^N S_j(0)$. The standard optimality condition now yields

$$C^i(t) = I^i \left(y^i \eta^i(t) e^{\delta t} \right) = \left(y^i \eta^i(t) e^{\delta t} \right)^{-1/\gamma},$$

where $I^i(\cdot)$ is the inverse marginal utility function of agent i and y^i is the Lagrange multiplier that solves the following budget constraint:

$$E^i \left[\int_0^T \eta^i(t) I^i \left(y^i \eta^i(t) e^{\delta t} \right) dt \right] = W^i(0).$$

The equilibrium allocations can be characterized by solving the optimization problem of a representative agent whose utility function is a weighted sum of the two agents' utilities,

$$U(C(t), \phi(t)) \equiv \max_{C^A + C^B = C} \left\{ U^A(C^A(t)) + \phi(t) U^B(C^B(t)) \right\}, \tag{17}$$

where the weight ϕ is stochastic and is driven by the difference in beliefs (see Equation (4)).[16] Hence the planner's problem (under $\mathbb{P}^A$, the pessimist's probability measure) is as follows:

$$J = \max_{C^A, C^B} E^A \left[\int_0^T e^{-\delta t} \left(\frac{C^A(t)^{1-\gamma}}{1-\gamma} + \phi(t) \frac{C^B(t)^{1-\gamma}}{1-\gamma} \right) dt \right] \quad \text{s.t.} \quad C^A(t) + C^B(t) = C(t). \tag{18}$$

The equilibrium consumption allocations are obtained from the first-order condition of the representative agent's problem while using individual agents' optimality conditions as just described.

Proposition 1 *Equilibrium consumption allocations are*

$$C^A(t) = \frac{1}{1 + \phi(t)^{1/\gamma}} C(t) \quad and \quad C^B(t) = \frac{\phi(t)^{1/\gamma}}{1 + \phi(t)^{1/\gamma}} C(t), \tag{19}$$

and investors' state price densities are

$$\eta^A(t) = e^{-\delta t} \frac{(1 + \phi(t)^{1/\gamma})^\gamma}{y^A C(t)^\gamma} \quad and \quad \eta^B(t) = \eta^A(t) \frac{\phi(0)}{\phi(t)} = e^{-\delta t} \frac{(1 + \phi(t)^{1/\gamma})^\gamma}{y^B C(t)^\gamma \phi(t)}. \tag{20}$$

Here $\phi(0)$ solves either agent's individual budget constraint,[17] and the stochastic weighting process $\phi(t) = y^A \eta^A(t) / y^B \eta^B(t)$ follows the dynamics given in Equation (4) with jump intensity $\lambda_c^A(t)$.

Proposition 1 characterizes the dependence of individual state price densities and consumption policies on C and ϕ, which represent aggregate endowment and belief disagreement risk in the economy,

---

[15]The market prices of diffusion and jump risk are not agent specific if the market is complete, since the agents have to agree on the observed price paths, see e.g. Dieckmann (2011). Completeness of the market is discussed in Section 3.3.

[16]The approach to formulating a representative agent problem with state-dependent weight was introduced by Cuoco and He (1994); more recent examples can be found in Basak and Cuoco (1998) and Buraschi and Jiltsov (2006). In a complete markets setting with heterogeneous beliefs the weight is stochastic and equal to the Radon–Nikodym derivative $\phi(t) = d\mathbb{P}^B / d\mathbb{P}^A$.

[17]The budget constraints of agents determine only the ratio y^A/y^B. I set $y^A = U'(C(0), \phi(0))$ without loss of generality, so that $\eta^A(0) = \eta^B(0) = 1$.

respectively. In this N-trees setting, the aggregate endowment C depends on the exogenous single dividend growth processes and on the dividend shares. Under homogeneous beliefs, the disagreement risk vanishes because ϕ is constant and depends only on initial wealth: $\phi = (x_s^B/x_s^A)^\gamma$. This means that, under homogeneous beliefs, the investors who are initially more wealthy consume more in all future states and times. In contrast, consumption differences can change sign if agents are heterogeneous. Namely, if a systemic disaster occurs, the consumption share of the pessimist (agent A) increases as ϕ jumps down.

For convenience, I define explicitly the consumption shares of the pessimistic and optimistic agents as

$$c^A(t) \equiv \frac{C^A(t)}{C(t)} \quad \text{and} \quad c^B(t) \equiv \frac{C^B(t)}{C(t)},$$

respectively. These shares will drive the market prices of risk and risk premia.

3.3 Price processes and market completeness

Assume the existence of a capital market that allows agents to share risk and finance consumption. The market consists of N risky assets with price vector $\mathbf{S}(t) = (S_1(t), S_2(t), \ldots, S_N(t))'$, each in unit net supply, as well as a riskless asset of price $B(t)$ in zero net supply. Then, for $j = 1, \ldots, N$, the price process dynamics are as follows:

$$\frac{dS_j(t) + D_j(t)dt}{S_j(t)} = \mu_{S_j}(t)dt + \boldsymbol{\sigma}_{S_j}(t)[d\mathbf{W}_t' \ dW_t^X]' + \mathbf{k}_{S_j}(t)d\mathbf{N}_t + k_{S_j}^c(t)dN_{ct}, \tag{21}$$

$$\frac{dB(t)}{B(t)} = r(t)dt. \tag{22}$$

To simplify notation, let me define the $N \times 1$ vector $\boldsymbol{\mu_S}$ of expected returns in normal times, the $N \times N+1$ matrix $\boldsymbol{\sigma_S}$ of diffusion volatilities, the $N \times N$ matrix $\boldsymbol{k_S}$ of return jump sizes related to idiosyncratic jumps, and the $N \times 1$ vector $\boldsymbol{k_S^c}$ of return jump sizes related to systemic jumps

$$\boldsymbol{\mu_S} = \begin{bmatrix} \mu_{S_1} \\ \mu_{S_2} \\ \vdots \\ \mu_{S_N} \end{bmatrix}, \quad \boldsymbol{\sigma_S} = \begin{bmatrix} \boldsymbol{\sigma}_{S_1} \\ \boldsymbol{\sigma}_{S_2} \\ \vdots \\ \boldsymbol{\sigma}_{S_N} \end{bmatrix}, \quad \boldsymbol{k_S} = \begin{bmatrix} \boldsymbol{k}_{S_1} \\ \boldsymbol{k}_{S_2} \\ \vdots \\ \boldsymbol{k}_{S_N} \end{bmatrix}, \boldsymbol{k_S^c} = \begin{bmatrix} k_{S_1}^c \\ k_{S_2}^c \\ \vdots \\ k_{S_N}^c \end{bmatrix}. \tag{23}$$

All the vectors and matrices in (23), as well as the riskless rate r, are determined endogenously in equilibrium. However, with only N risky securities the market is incomplete, since they only span the uncertainty driven by the Brownian motions.[18] Hence I assume agents can also trade in $N+1$ rare event insurance products $P_j(t)$, $j = 1, \ldots, N$ and $P_c(t)$, which are in zero-net supply, do not pay dividends, and have price processes

$$\frac{dP_j(t)}{P_j(t)} = \mu_{p_j}(t)dt + k_{p_j}(t)dN_{jt}, \tag{24}$$

$$\frac{dP_c(t)}{P_c(t)} = \mu_{p_c}(t)dt + k_{p_c}(t)dN_{ct}, \tag{25}$$

---

[18]More precisely, there are $N+1$ Brownian shocks in the economy; however, the risk of changes in the disaster probability (i.e., shocks to W_t^X) are not priced in the power utility setting although they would be if agents had recursive preferences. See also Wachter (2013).

where μ_{p_j} and μ_{p_c} are determined in equilibrium, whereas jump sizes can be freely chosen and need only be different from zero in order to complete the market. These assets can be interpreted as insurance products against rare event risk because they do not contain any continuous source of uncertainty. The buyer of asset P_j, $j = 1, \ldots, N, c$, is rewarded in the amount μ_{p_j} every moment of time, but runs the risk that the asset's value drops to $(1 + k_{p_j})P_j$ when the corresponding Poisson process N_{jt} jumps. Therefore, selling assets P_j, $j = 1, \ldots, N$, provides insurance against idiosyncratic jumps, P_c is a form of insurance against systemic disasters.[19] In general, any set of $N + 1$ assets spanning all jump components would complete the market, but the choice of disaster insurances is the most appealing since it isolates the impact of the different rare events.

3.4 Market prices of risk

Market prices of risk are obtained by applying Itô's lemma to Equation (20) and then comparing the resulting dynamics with Equation (16). They are summarized in the following proposition.

Proposition 2 *The market prices of normal economic risk, both risk-neutral rare event intensities, and the short rate are given by*

$$\theta_j(t) = \gamma s_j \sigma, \tag{26}$$

$$\lambda_j^Q = \lambda(s_j k + 1)^{-\gamma}, \tag{27}$$

$$\lambda_c^Q(t) = \left(c^A(t)\lambda_c^A(t)^{1/\gamma} + c^B(t)\lambda_c^B(t)^{1/\gamma} \right)^\gamma (k+1)^{-\gamma}, \tag{28}$$

$$r(t) = \delta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sum_{j=1}^N s_j^2\sigma^2 - c^B(t)(\lambda_c^A(t) - \lambda_c^B(t)) + \sum_{j=1}^N (\lambda - \lambda_j^Q(t)) + (\lambda_c^A(t) - \lambda_c^Q(t)). \tag{29}$$

The market price of economic risk has the standard solution as extended to the case of N trees. The risk-neutral intensity λ_j^Q of an idiosyncratic jump in the dividend process D_j depends only on the dividend share of asset j given that agents agree on the physical idiosyncratic jump intensities. For any dividend share distribution, the idiosyncratic jump risk premia $\lambda_j^Q(t)/\lambda$ are constant and always greater than 1, and they tend to unity as s_j approaches zero. Thus the risk of idiosyncratic jumps in small assets is not priced, and in general the price associated with idiosyncratic jump risk is small when the number of stocks in the economy, N, is large. The risk-neutral common disaster frequency λ_c^Q is a nonlinear function of the two agents common jump intensities weighted by their consumption shares; it could be smaller than the physical intensity when the optimist's consumption share is large, leading to a systemic jump premium of less than 1. The riskless interest rate follows the standard expression in Lucas economies with multiple trees (see e.g. Cochrane, Longstaff, and Santa-Clara (2008)), with the addition of three components related to disagreement and jump premia. The equilibrium short rate is generally decreasing with the consumption share of the pessimistic agent A, as in the aftermath of a disaster.

---

[19]Catastrophe bonds can be viewed as the real-world counterpart to these theoretical securities.

Using Proposition 2, it is possible to derive explicitly the risk premia on the risky assets once the volatilities and jump sizes of the stock price processes (i.e., $\boldsymbol{\sigma}_S(t)$, $\boldsymbol{k}_S(t)$ and $\boldsymbol{k}_S^c(t)$) are known. This can be done by applying Itô's lemma to the stock prices. For integer risk aversion γ, the resulting equation can be solved in semi-closed-form as summarized in the next proposition.

Proposition 3 *The price of stock j is given by*

$$S_j(t) = E_t^A \left[\int_t^T \frac{\eta^A(s)}{\eta^A(t)} D_j(s) \ ds \right] = D_j(t) g_j(\phi(t), X(t), \boldsymbol{u}(t), t). \tag{30}$$

Here the price-dividend ratio g_j depends on time t, on the stochastic weighting process ϕ, on the state variable X that drives time-varying systemic disaster intensity, and on the dividend share distribution through the $(N-1)$-dimensional state variable $\boldsymbol{u}$:

$$g_j(\phi, X, \boldsymbol{u}, t) = e^{-\gamma \sum_{j=2}^N u_j/N} \left(1 + e^{u_2} + \cdots + e^{u_N}\right)^\gamma \sum_{k=0}^\gamma a_k(\phi) \int \mathcal{F}_\gamma^N(\mathbf{z}) e^{i\boldsymbol{u}'\mathbf{z}} b_{jk}(X, t, \mathbf{z}) \ d\mathbf{z}, \tag{31}$$

where the integral is evaluated on $\mathbb{R}^{N-1}$, $\mathcal{F}_\gamma^N(\mathbf{z})$ is given by Equation (77) in Appendix A.3 and

$$a_k(\phi) = \binom{\gamma}{k} \frac{\phi(t)^{k/\gamma}}{(1 + \phi(t)^{1/\gamma})^\gamma},$$

$$b_{jk}(X, t, \mathbf{z}) = \int_t^T e^{(\tau-t)\left[-\delta + (\mu - \frac{1}{2}\sigma^2)\mathbf{1}_N'(\mathbf{e}_j - \gamma/N + i\mathbf{U}'\mathbf{z}) + \frac{1}{2}\sigma^2(\mathbf{e}_j - \gamma/N + i\mathbf{U}'\mathbf{z})'(\mathbf{e}_j - \gamma/N + i\mathbf{U}'\mathbf{z})\right] + \alpha_{0,k}^N(\tau-t) + \alpha_{2,k}^N(\tau-t)X(t)} \ d\tau.$$

Here $\mathbf{e}_j$ is the N-vector with a 1 in the jth entry and 0s elsewhere, $\mathbf{1}_N$ is an N-dimensional vector of 1s and $\alpha_{0,k}^N(\tau)$ and $\alpha_{2,k}^N(\tau)$ satisfy the system of Riccati equations given in Appendix A.3.

Semi-closed-form expressions[20] for diffusion volatilities and jump sizes of stock j's return process follow after application of Itô's lemma for jump-diffusion processes, using dividend growth, stochastic weight process, dividend shares and exogenous state variable dynamics in Equations (1), (4), (10) and (3), respectively:

$$\boldsymbol{\sigma}_{S_j}(t) = \left[\sigma \left(\mathbf{e}_j' + \frac{g_{j\mathbf{u}}'}{g_j} \mathbf{U} \right) \quad \frac{g_{jX}}{g_j} \sigma_X \sqrt{X(t)} \right], \quad k_{S_j}^c(t) = (k+1) \frac{g_j \left(\phi \frac{\beta^B}{\beta^A}, X, \mathbf{u}, t \right)}{g_j(\phi, X, \mathbf{u}, t)} - 1. \tag{32}$$

The ith component of vector $\boldsymbol{k}_{S_j}(t)$ is given by

$$k_{S_j, i}(t) = \begin{cases} \frac{g_j(\phi, X, \mathbf{u} + \ln(k+1)\mathbf{U}\mathbf{e}_i, t)}{g_j(\phi, X, \mathbf{u}, t)} - 1 & \text{if} \quad i \neq j, \\ (k+1) \frac{g_j(\phi, X, \mathbf{u} + \ln(k+1)\mathbf{U}\mathbf{e}_i, t)}{g_j(\phi, X, \mathbf{u}, t)} - 1 & \text{if} \quad i = j, \end{cases} \tag{33}$$

where $g_{j\mathbf{u}}$ and g_{jX} are the derivatives of the price-dividend ratio g_j with respect to $\mathbf{u}$ and X, respectively, which can also be obtained in semi-closed form. Time-varying disaster risk and disagreement endogenously generate time variation in the diffusion volatilities and jump sizes of stock returns, even if the parameters in the dividend growth processes are constant.

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[20]Up to the solution of the ordinary differential equations for $\alpha_{0,k}(\tau)$ and $\alpha_{2,k}(\tau)$, which is easily obtained numerically after evaluating an $(N-1)$-dimensional integral that is well-behaved but can be computationally intensive for large N.

4 Results and Analysis

In this section I study the properties of the risk premia and other asset pricing implications of the model presented in Section 3.

Instead of attempting to estimate the model, I analyze its main qualitative implications and the mechanisms behind them by means of a simple numerical illustration for a symmetric economy with two stocks, $N = 2$. In the baseline calibration, dividend growth processes have a drift $\mu = 2.5\%$ and a diffusion volatility $\sigma = 5\%$.[21] Rare events have an impact of $k = -0.41$, consistently with the estimates reported in Dieckmann and Gallmeyer (2005) and in Barro (2006). Idiosyncratic jumps have a constant intensity $\lambda = 1\%$ and systemic jumps occur with a long-term frequency $\beta^A = 1\%$, so jumps in individual dividend processes occur on average each fifty years. The optimistic agent believes that the long-term mean of the frequency of systemic disasters is smaller than does the pessimistic agent; that is, $\beta^B < \beta^A$. In most cases I fix $\beta^B = 0.01\%$ but various levels of the difference $\beta^A - \beta^B$ are also considered. The two agents have the same CRRA preferences along with a time horizon T of 50 years, a time preference rate $\delta = 4\%$, and a risk aversion parameter $\gamma = 4$. The parameters of the X process are $\varphi = 0.142$ and $\sigma_X = 0.05$. Preference parameters are taken from Chen, Joslin, and Tran (2012), while the parameters in the X process are chosen to match the properties of Chen, Joslin, and Tran (2012)'s calibrated time-varying disaster intensity. Model parameters are summarized in Table 1.

4.1 Equity and variance risk premia

From agent i's perspective, the risk premium for any security is defined as the difference between the expected return under $\mathbb{P}^i$ and under the risk-neutral measure $\mathbb{Q}$. I report risk premia relative to agent A's beliefs, $\mathbb{P}^A$. Define the cum-dividend instantaneous return of stock j as

$$dR_{jt} = \frac{dS_j(t) + D_j(t)dt}{S_j(t)}.$$

The instantaneous conditional equity risk premium of the individual stock j, ERP_j, is thus

$$
\begin{aligned}
ERP_{jt} &= E_t^A(dR_{jt}) - E_t^Q(dR_{jt}) \\
&= \gamma\sigma^2\left(\mathbf{e}_j' + \frac{g_j'\mathbf{u}}{g_j}\mathbf{U}\right)\mathbf{s} + \sum_{i=1}^N k_{S_j,i}(t)(\lambda - \lambda_i^Q(t)) + k_{S_j}^c(t)(\lambda_c^A(t) - \lambda_c^Q(t)) \\
&= \gamma\sigma^2\left(\mathbf{e}_j' + \frac{g_j'\mathbf{u}}{g_j}\mathbf{U}\right)\mathbf{s} - \lambda\sum_{i=1}^N k_{S_j,i}(t)(JP_{it} - 1) - \lambda_c^A(t)k_{S_j}^c(t)(JP_{ct} - 1),
\end{aligned} \tag{34}
$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_N)'$ is the vector of dividend shares, $JP_{it} = \lambda_i^Q(t)/\lambda$ is the jump premium related to an idiosyncratic jump in the dividend growth of asset i, and $JP_{ct} = \lambda_c^Q(t)/\lambda_c^A(t)$ is the jump premium related to a common jump. The first term in (34) is the compensation for diffusion risk; the other terms represent a premium for bearing idiosyncratic and systemic disaster risk, respectively.

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[21] The values for the diffusion component of the dividend dynamics, μ and σ, are within the ranges considered in the literature. See, among others, Campbell (2003) and Cochrane, Longstaff, and Santa-Clara (2008).

As mentioned in Section 3.4, the jump premium for idiosyncratic event risk, $JP_{it} = (s_i k + 1)^{-\gamma}$, is always greater than 1 and it is also close to 1 for small stocks (see right panel of Figure 2). We can use Equation (28) to write the jump premium for systemic event risk as

$$\frac{\lambda_c^Q(t)}{\lambda_c^A(t)} = \left[\frac{1 + \left(\phi(t)\frac{\beta^B}{\beta^A} \right)^{1/\gamma}}{(1 + \phi(t)^{1/\gamma})} \right]^\gamma (k+1)^{-\gamma} = (k_{C^A}(t) + 1)^{-\gamma}, \tag{35}$$

where

$$k_{C^A}(t) = \frac{(1 + \phi(t)^{1/\gamma})(k+1)}{1 + \left(\phi(t)\frac{\beta^B}{\beta^A} \right)^{1/\gamma}} - 1$$

is the size of the jump in equilibrium consumption of agent A in response to a systemic disaster. This jump size varies depending on the level of disagreement and the consumption share distribution, due to risk sharing between agents. Since agent B (the optimist) thinks systemic disasters are highly unlikely, he is willing to give up consumption in future systemic disaster states in exchange for higher consumption in all other future states. This mechanism reduces the consumption loss of agent A in the event of a systemic disaster and lowers the corresponding jump risk premium. The more wealth the optimist has, the more disaster insurance he is able to sell. So when the wealth share of the optimist is high, consumption of agent A can even increase at a disaster. That scenario would lead to a jump premium lower than 1 (see left panel of Figure 2)—in other words, to a risk-neutral intensity λ_c^Q lower than the physical intensity λ_c^A.[22] A higher level of relative risk aversion γ would lead to a much faster rise in the systemic jump premium, although the qualitative implications would remain unchanged.

Figure 1 shows the conditional instantaneous equity premium of stock 1 at time $t = 0$ and its components, as a function of the dividend share s_1, for two possible values of the initial wealth share of the pessimistic agent A ($c^A = 0.1$ in the left panel and $c^A = 0.9$ in the right panel). The equity premium is first slightly decreasing and then increasing in the dividend share of the asset; a pattern that is due to the behavior of the compensation for diffusion risk (see Martin (2013)) and to the fact that the overall premium for idiosyncratic risk is lower for intermediate values of the dividend share. Note that the compensation for diffusion and idiosyncratic rare event risk does not change with the consumption share of the two agents, since they disagree only with respect to the systemic disaster intensity. The contribution to stock 1's equity premium of its own idiosyncratic jump risk starts at zero but increases substantially with its dividend share, as the asset becomes more *systemic*. The compensation due to idiosyncratic rare event risk in asset 2's dividends is small unless the second stock contributes to a large fraction of aggregate consumption. On the other hand, the component of asset 1's equity premium that

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[22]More precisely, the systemic jump premium is less than 1 when the ratio of the consumption shares is large:

$$\frac{c^B(t)}{c^A(t)} > \frac{-k}{k + 1 - \left(\frac{\beta^B}{\beta^A} \right)^{1/\gamma}}$$

if the disagreement is large enough, that is, if

$$\frac{\beta^B}{\beta^A} < (k+1)^\gamma.$$

In the calibration this condition is satisfied when the consumption share of the optimist, $c^B(t)$, is at least 60%.

is due to systemic rare event risk is basically flat with the dividend share but depends on disagreement risk and reflects risk sharing between agents. The compensation for systemic jump risk is negative for small consumption shares of the pessimist but increases rapidly, and for large values of c^A that compensation accounts for a large fraction of the individual equity premium (mainly when dividends are evenly distributed between the two stocks). This effect is primarily driven by the jump premium for systemic disasters, JP_c, in the left panel of Figure 2. Besides the jump risk premia, the equity premium is also a function of the jump sizes of stock returns $\boldsymbol{k}_s$ and $\boldsymbol{k}_s^c$, which depend on the dividend loss and on changes in the price-dividend ratios, as shown in Equations (33) and (32). The left panel of Figure 3 plots the jump size in the return of a stock at a systemic disaster, $k_{S_j}^c$, as a function of the consumption share of the pessimistic agent, c^A, for a small stock ($s = 0.1$) and a large stock ($s = 0.9$). Under CRRA utility, the drop in the risk-free rate following a systemic disaster can dominate the effect of a rising risk premium, which would lead to a higher price-dividend ratio. That higher price-dividend ratio partially offsets the drop in dividends, making the return less sensitive to systemic disasters. The variation of systemic jump size in stock returns with the pessimist's consumption share is stronger for a small stock (blue line in the figure) and depends crucially on the assumption of difference in beliefs. In fact, if there is no disagreement then the systemic jump size $k_{S_j}^c$ is constant and equal to the loss in dividend growth k (see the black dotted line in the left panel of Figure 3). The right panel displays the jump size in stock 1's return at an idiosyncratic jump in its dividend growth process (blue line), $k_{S_1,1}$, and in the dividend growth process of the second asset (red line), $k_{S_1,2}$. Observe that $k_{S_1,2}$ can become slightly positive for large s_1, which means that idiosyncratic jump risk in the dividend growth of the small stock can have a negative effect on the equity premium of the large stock. This effect can be interpreted as a flight to safety from the small to the large stock.

In the same way, the instantaneous variance risk premium of stock j, VRP_j, can be computed as the difference between objective and risk-neutral expectations of the return variance:

$$
\begin{aligned}
VRP_{jt} &= E_t^A[(dR_{jt})^2] - E_t^Q[(dR_{jt})^2] \\
&= \sum_{i=1}^{N} k_{S_j,i}(t)^2(\lambda - \lambda_i^Q(t)) + k_{S_j}^c(t)^2(\lambda_c^A(t) - \lambda_c^Q(t)) \\
&= -\lambda \sum_{i=1}^{N} k_{S_j,i}(t)^2(JP_{it} - 1) - \lambda_c^A(t)k_{S_j}^c(t)^2(JP_{ct} - 1).
\end{aligned}
\tag{36}
$$

Given the assumption of constant dividend growth volatilities, the variance risk premium depends only on the jump risk components. Yet empirical evidence reported in Bollerslev and Todorov (2011) and Ait-Sahalia, Karaman, and Mancini (2012) shows that compensation for rare events actually accounts for a large fraction of variance risk premia. The instantaneous variance premium VRP_j is usually negative, as expected, but it can become positive when $JP_{ct} < 1$ and large enough to balance out the contribution of the idiosyncratic jump components, which is always negative (as discussed previously; see Figure 4). The variance risk premium is negatively related to the systemic jump premium: it decreases with agent A's consumption share, and it is either decreasing or hump-shaped with respect to a stock's dividend

share (depending on the value of the calibrated parameters). As for the individual equity premium, the compensation due to idiosyncratic rare event risk in asset 2's dividends is nearly zero; however, the contribution of idiosyncratic rare event risk in its own dividend process is increasing (in absolute value) in the dividend share.

The model relates the correlation between individual variance premia to the systemic rare event risk. This systemic component is stronger when the consumption share of the pessimist is higher. In the case of a two-stocks economy, the average model-implied correlation between variance premia ranges from -0.4 (when the consumption share of agent A is 10%) to about 0.75 (when the pessimist consumes 90% of the aggregate dividend).

Now let me define the instantaneous return on the stock market index as the weighted sum of all individual asset returns:[23]

$$dR_t = \sum_{j=1}^{N} s_j dR_{jt}. \tag{37}$$

The instantaneous equity premium on the index is then

$$
\begin{aligned}
ERP_t &= \sum_{j=1}^{N} s_j \; ERP_{jt} \\
&= \gamma \sigma^2 \mathbf{s}' \left(\mathbf{I}_N + \frac{\mathbf{g_u}'}{\mathbf{g}} \mathbf{U} \right) \mathbf{s} + \mathbf{s}' \mathbf{k}_s(t)(\boldsymbol{\lambda} - \boldsymbol{\lambda}^Q(t)) + \mathbf{s}' \mathbf{k}_s^c(t)(\lambda_c^A(t) - \lambda_c^Q(t)),
\end{aligned}
\tag{38}
$$

where $\mathbf{I}_N$ is the identity matrix of dimension N, $\boldsymbol{\lambda}^Q = (\lambda_1^Q, \lambda_2^Q, \ldots, \lambda_N^Q)'$, $\frac{\mathbf{g_u}'}{\mathbf{g}} = \left(\frac{g_{1u}}{g_1}, \frac{g_{2u}}{g_2}, \ldots, \frac{g_{Nu}}{g_N} \right)'$, and the instantaneous index variance risk premium is given by

$$
\begin{aligned}
VRP_t &= \sum_{j=1}^{N} s_j^2 \; VRP_{jt} + \sum_{j=1}^{N} \sum_{i \neq j} s_j s_i \; CRP_{jit} \\
&= \mathbf{s}' \left[\mathbf{k}_s(t) \; \text{diag}(\boldsymbol{\lambda} - \boldsymbol{\lambda}^Q(t)) \mathbf{k}_s(t)' + \mathbf{k}_s^c(t) \mathbf{k}_s^c(t)'(\lambda_c^A(t) - \lambda_c^Q(t)) \right] \mathbf{s}.
\end{aligned}
\tag{39}
$$

Here $CRP_{jit} = E_t^A[dR_{jt} dR_{it}] - E_t^Q[dR_{jt} dR_{it}]$ is the premium associated with the covariance between returns of assets j and i, and $\text{diag}(\boldsymbol{\lambda} - \boldsymbol{\lambda}^Q(t))$ is an $N \times N$ matrix with the elements of the N-vector $\boldsymbol{\lambda} - \boldsymbol{\lambda}^Q(t)$ on the diagonal and with 0s elsewhere.

Figure 5 plots the instantaneous equity (upper panels) and variance (lower panels) risk premium of the market, under agent A's beliefs, as a function of the dividend share of asset 1, s_1, and their decomposition in terms of individual equity and variance premia for different values of the consumption share of the pessimistic agent. The market equity premium increases with the consumption share of the pessimist, and it is lower when the two assets contribute in the same way to the aggregate dividend because the equity premium of individual stocks grows more than linearly with the dividend share. The same reasoning holds for the absolute value of the aggregate variance premium—which includes, however, an additional component reflecting the priced covariance between stock returns. The covariance premium can contribute to a large portion of the aggregate variance premium when the economy is dominated

---

[23]The stock market index can also be viewed as a claim on the aggregate endowment $C = D_1 + \cdots + D_N$.

by the pessimistic agent, mostly when the number of assets increases and they are relatively evenly distributed.

Apart from the aggregate variance premium's dependence on the relative dividend and consumption shares, its dynamic properties are worth examining. I simulate 30-year paths of the variance risk premium at a monthly frequency from the model while using calibrated parameters for different values of the initial wealth share of the pessimistic agent, c^A. Table 2 shows that, as the consumption share of the pessimist increases, the VRP is both larger (in absolute value) and more volatile. A systemic disaster induces an upward jump in the consumption share of the pessimist. That leads to a downward jump in the variance risk premium, which is then followed by more negative and volatile premia. Despite the setting's simplicity, the dynamics of model-implied premia resembles the behavior of observed variance risk premia (see Section 5 and Figure 13), in which periods of low and smooth premia seem to be followed by larger and more volatile values. Empirically a regime switch often corresponds to the beginning of a crisis, so it could be linked to a systemic jump in the endowment process.

4.2 Stock return correlation and correlation risk premium

From Equation (21), the instantaneous conditional correlation between returns of stock i and stock j is given by

$$Corr_t^A(dR_{it}, dR_{jt}) = \frac{\boldsymbol{\sigma}_{S_i}(t)\boldsymbol{\sigma}'_{S_j}(t) + \mathbf{k}_{S_i}(t)\mathbf{k}'_{S_j}(t)\lambda + k_{S_i}^c(t)k_{S_j}^c(t)\lambda_c^A}{\sqrt{(\boldsymbol{\sigma}_{S_i}(t)\boldsymbol{\sigma}'_{S_i}(t) + \mathbf{k}_{S_i}(t)\mathbf{k}'_{S_i}(t)\lambda + k_{S_i}^c(t)^2\lambda_c^A)(\boldsymbol{\sigma}_{S_j}(t)\boldsymbol{\sigma}'_{S_j}(t) + \mathbf{k}_{S_j}(t)\mathbf{k}'_{S_j}(t)\lambda + k_{S_i}^c(t)^2\lambda_c^A)}}. \tag{40}$$

The first panel in Figure 6 shows the conditional stock return correlation in a symmetric economy with $N = 2$ stocks as a function of the first tree's dividend share s_1 and the pessimistic investor's consumption share c^A while using the model parameters in Table 1. The other panels in Figure 6 display the same correlation for special cases of the model. The second panel considers the case of no disagreement ($\beta^A = \beta^B = 0.01$), in the fourth panel I assume there are no idiosyncratic disasters ($\lambda = 0$), and the third panel combines these last two cases. Comparing the first and second (or the third and fourth) panels reveals that disagreement reduces stock return correlation on average and in particular when risk sharing is stronger—that is, when the consumption shares of the two agents are similar. The possibility of idiosyncratic disasters in the dividend growth processes also reduces the average correlation (compare the first and third panels of Figure 6), albeit mainly when dividend shares are relatively evenly distributed. Overall, however, the correlation under the pessimistic agent's objective measure is relatively flat: it has values between 35% and 43% and an average across all states of about 39%.

The correlation risk premium is defined as the difference between the instantaneous conditional correlation computed under the physical and the risk-neutral measure,

$$CorrRP_{ij,t} = Corr_t^A(dR_{it}, dR_{jt}) - Corr_t^Q(dR_{it}, dR_{jt}). \tag{41}$$

Here the risk-neutral correlation is computed as in (40) but using the risk-neutral idiosyncratic and systemic rare event intensities $\boldsymbol{\lambda}^Q$ and λ_c^Q, respectively; see Figure 7. The model-implied risk-neutral

correlation (first panel) increases substantially with the consumption share of the pessimistic agent and ranges approximately between 11% and 75%. This result is consistent with the empirical findings of Driessen, Maenhout, and Vilkov (2012), who show that the implied correlation for the $S\&P500$ is highly countercyclical and fluctuates between 0.2 and 0.8 for the period 1996–2010. The other panels in Figure 7 show that the dynamics of the risk-neutral correlation is almost entirely driven by disagreement between agents about the probability of a systemic disaster. The average risk-neutral correlation for the full model is about 46%, which corresponds to an average instantaneous correlation risk premium of about -7%; this value, too, is consistent with the empirical findings reported by Driessen, Maenhout, and Vilkov (2012). However, the model-implied correlation premium (see Figure 8) can be much larger in absolute value when the pessimist accounts for a large part of the aggregate consumption, and it can also become positive when the pessimist's consumption share is relatively low—mainly when the dividend shares of the two assets are similar.

4.3 Relation between the equity and the variance risk premium

Comparing the expressions for the variance premium (Equations (36) and (39)) with those for the equity premium (Equations (34) and (38)) shows that rare event risk implies a tight link between the two, both for the market and for the cross section of stock returns. This link provides our basic intuition for the role of the variance premium in predicting future excess returns, which is consistent with Bollerslev, Tauchen, and Zhou (2009)'s empirical finding that aggregate variance risk premium can explain a nontrivial fraction of the time-series variation in post-1990 aggregate stock market returns. Premia that are high (in absolute value) predict high future returns—though mainly over short horizons, when the compensation for rare events accounts for a large portion of the empirical equity and variance risk premia (see e.g. Bollerslev and Todorov (2011) and Ait-Sahalia, Karaman, and Mancini (2012)). Yet standard predictive regressions imply an unconditionally linear relation between equity and variance risk premia, whereas the model's relation is conditional on the information set at time t. The idiosyncratic and systemic event risk components of the equity and variance risk premium are linearly related, but the regression coefficients are stochastic and given by the inverse of the corresponding jump size. Depending on which of the jump risk components dominates, the relation can be either weaker or stronger. The importance of the idiosyncratic and systemic rare event risk contribution to the risk premia, as well as the jump sizes in stock returns, are both functions of the asset's dividend share and the agents' consumption share. Thus, time variation in share distributions leads to a time-varying relation between equity and variance risk premia, both at an aggregate level and for individual stocks. For individual stocks, however, empirical estimates of the variance premium are noisy owing to lack of reliable high-frequency data for computing the realized variance. Moreover, there is evidence of a large systematic component in the cross section of variance risk premia (see e.g. Carr and Wu (2009)). Hence this paper also explores the relation between the instantaneous equity premium of individual stocks and the market's variance risk premium.

To develop a better understanding of the model implications that concern the predictive power of

aggregate variance premium for market and individual stock excess returns, I run regressions on simulated data. This involves simulating 30 years of monthly excess stock and market returns and the instantaneous variance risk premium from the model in Section 3 while assuming a symmetric economy with $N = 2$ stocks and using the baseline model parameters. The purpose of these simulations is to investigate the model's qualitative implications for the interaction between the aggregate variance premium and the excess stock and market returns. This is a natural step between the model and the empirical evidence presented in Section 5. In addition to the monthly return horizon, I also consider multi-period return regressions of the form

$$r_{i,t+h}^e = \alpha_i + \beta_i \ VRP_t + \varepsilon_{i,t+h},$$

where $r_{i,t+h}^e$ is the simulated log excess return of the two stocks ($i = 1, 2$) or of the market ($i = M$) and where h is the return horizon in months. The excess return is given as an annualized percentage and the variance premium is given as a monthly squared percentage for consistency with the literature (see e.g. Bollerslev, Tauchen, and Zhou (2009) and Drechsler (2011)). Table 3 reports the average regression coefficient and adjusted R^2 (with standard errors in parentheses) for the market at horizons $h = 1$, 6, and 12 months and for different values of the initial consumption share of the pessimistic agent, $c^A = 0.1$, 0.5 and 0.9. The predictive coefficient is generally negative. The predictive power increases with the horizon and with the initial wealth share of the pessimistic agent, which is also associated with larger (absolute) values of the variance risk premium and of its volatility (again, see Table 2). Note that the average estimated regression coefficient is quite close to what is found in the data (see Section 5), even if the model is not estimated or calibrated to match the observed VRP moments. Figure 9 presents box plots of the predictive regression coefficients (upper panel) and of the adjusted R^2 (lower panel) at the 6-month horizon. The regression coefficient is significantly different from zero only when the pessimist holds a large fraction of the aggregate endowment.

Turning now to the cross section, the first two panels of Table 4 display results of the same predictive regressions for the two individual stocks in the economy. Initially, the small stock (Panel A) has dividend share $s_1 = 0.1$ and the big stock (Panel B) has a share $s_2 = 0.9$. The regression coefficient for the small stock is often positive and not significant. The reason is that, with only two stocks in the economy, the fear of idiosyncratic disasters in the dividend growth of the large stock has a strong effect on the equity premium of the small stock; the corresponding jump size in the small stock return, $k_{S_1,2}(t)$, can be positive and thereby lead to a weak positive relation between the equity premium of the small stock and the market variance risk premium. This effect holds also after eliminating idiosyncratic jump risk ($\lambda = 0$) for small values of the consumption share of the pessimist, because in that case the variance risk premium of the small stock is negatively correlated with the market variance premium. In contrast, the predictive regression results for the large stock (Panel B) are in line with those discussed for the market return regression, since the big stock contributes to a large fraction of the aggregate dividend.

An economy consisting only of two stocks, one of which accounts for 90% of aggregate consumption, is clearly not realistic. It would be interesting to run cross-sectional predictive regressions for an economy

with many assets and relatively small values of the dividend share, since these features better characterize real-world markets. However, it is not computationally feasible to simulate the model for large N because the solution would require numerical evaluation of a high-dimensional integral at each time step. However, Section 4.4 investigates theoretically the special case of a large and diversified economy and demonstrates that, as N increases, the idiosyncratic jump premium contribution the both equity and variance risk premia vanishes and the aggregate variance risk premium becomes due almost entirely to a covariance premium. So in order to mimic the case of a large economy without the need to simulate it, I look at the predictive power of the simulated covariance risk premium for the excess return of individual stocks. Panels C and D of Table 4 display results of the regression

$$r^e_{i,t+h} = \alpha_i + \beta_i\ CRP_t + \varepsilon_{t+h},$$

where $r^e_{i,t+h}$ is the simulated log excess return of the small and the big stock and CRP is the covariance risk premium (in monthly squared percentage). For the small asset (Panel C), the predictive coefficient is negative and significantly different from zero. Its average value is similar across horizons and consumption shares of the pessimist, but the standard deviation of the regression coefficient decreases with c^A and so leads to high adjusted R^2 when the pessimist accounts for a large share of aggregate consumption. On average, the R^2 values are even higher than those reported in Table 3 for the aggregate market. At the 6-month horizon, for example, the average adjusted R^2 for the regression of excess small asset returns on the instantaneous covariance premium is almost 19%, as compared with a 14% R^2 for the regression of market excess returns on the variance premium. For the large asset (Panel D), results are much weaker. Regression coefficients are even positive for small values of the pessimist's consumption share yet become negative (but only marginally significant) for large c^A. Figure 9 presents box plots of the predictive regression coefficients (upper panel) and of the adjusted R^2 (lower panel) obtained by regressing simulated 6-month excess returns of a small stock (starting from $s = 0.1$; blue box plots) and a big stock (starting from $s = 0.9$; red box plots) on the simulated lagged instantaneous covariance risk premium for different levels of the initial share of consumption of the pessimistic agent, c^A. These results indicate that, in a relatively large economy, the forecasting power of the aggregate variance risk premium for future excess returns should be stronger for small stocks because their returns are more dependent on the compensation for systemic rare event risk (though mainly for large values of the consumption share of the pessimistic agent). This model prediction is tested empirically in Section 5.3.

4.4 The case of a large economy: $N \to \infty$

Let me now consider analytically the case in which the number of assets in the economy, N, approaches infinity and dividends are evenly distributed across assets; that is, $s_j = 1/N$ for $j = 1, \dots, N$. In this case, the premium for idiosyncratic risk in individual assets vanishes because $JP_{jt} = (k/N + 1)^{-\gamma}$ converges to unity for all j. Moreover, the diffusion component in the equity premium for stock j reduces to $\gamma\sigma^2/N$, which tends to zero as the number of assets N increases. Hence the expressions for the equity

risk premium of stock j and of the index in Equations (34) and (38) can be simplified as follows:

$$ERP_{jt} = -\lambda_c^A(t)k_{S_j}^c(t)(JP_{ct} - 1), \tag{42}$$

$$ERP_t = \gamma\sigma^2 - \lambda_c^A(t)k_{S_j}^c(t)(JP_{ct} - 1). \tag{43}$$

The equity premium is the same for any stock because in this special case, $k_S^c(t) \equiv k_{S_j}^c(t) = k_{S_i}^c(t)$ for all i and j. Furthermore, the market equity premium is equal to the equity premium of single stocks plus the standard (constant) compensation for diffusion risk, $\gamma\sigma^2$, that arises in economies where dividend growth follows a geometric Brownian motion. Note that even if stocks are negligibly small, they still earn a risk premium due to the presence of the systemic jump component, which does not depend on the dividend share (see the black dashed-dotted lines in Figure 1).

Similarly, the variance risk premium is the same for any individual stock,

$$VRP_{jt} = -\lambda_c^A(t)k_S^c(t)^2(JP_{ct} - 1), \tag{44}$$

and is equal to the premium for the covariance between stock i and stock j, $CRP_{jit} = VRP_{jt}$. Then by way of Equation (39), the variance premium for the market index becomes

$$\begin{aligned} VRP_t &= \sum_{j=1}^{N} s_j^2 \, VRP_{jt} + \sum_{j=1}^{N}\sum_{i\neq j} s_j s_i \, CRP_{jit} \\ &= \frac{VRP_{jt}}{N} + \frac{N-1}{N} \, CRP_{jit} \\ &= -\lambda_c^A(t)k_S^c(t)^2(JP_{ct} - 1), \end{aligned} \tag{45}$$

which is equal to the variance risk premium of any individual stock and also to the covariance premium. In particular, from the second line of Equation (45) it is evident that, as N increases, all the market variance risk premium is due to a premium for covariance. In accordance with this model-implied feature, Driessen, Maenhout, and Vilkov (2012) show empirically that the variance risk premium for the $S\&P500$ index can be largely attributed to the high price of correlation risk.

To clarify premia behavior in this special case as a function of the consumption share distribution, Figure 11 shows the equity premium of stock j and of the index, the systemic jump premium, and the index variance risk premium as functions of the consumption share of the pessimistic agent, c^A.[24] In this case the link between variance risk premia and excess stock returns, both for the index and for single stocks, is straightforward:

$$ERP_t = \gamma\sigma^2 + \frac{1}{k_s^c(t)} \, VRP_t, \tag{46}$$

$$ERP_{jt} = \frac{1}{k_s^c(t)} \, VRP_t. \tag{47}$$

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[24]Numerical results are obtained for the parameters in Table 1 and $N = 10$, which is not that large but already entails solving a 9-dimensional integral—even though in the special case of equal dividend shares, the expression for the price-dividend ratio is simpler than in the general case in Section 3 (see Appendix A.3.3). Nonetheless, already for $N = 8$ or 9 the results are nearly identical.

and it is linear conditionally on the information set at time t. In particular, the regression coefficient $1/k_s^c(t)$ depends only on the consumption share of the two agents (as shown in Figure 12). This relation is negative and stronger for large values of the consumption share of the pessimist; the maximum is around $c^A = 0.7$, above which the relation becomes weaker for extreme values of c^A.

For a large and diversified index such as the $S\&P500$, the model thus suggests a stronger predictive power of variance risk premium for future excess returns in periods during which pessimists have a relatively large consumption share—that is, in bad states of the economy, which are also generally linked to higher (absolute) values of the variance risk premium. I investigate this intuition empirically in Section 5.2.

4.5 Consumption share dynamics and survival

Agent survival is an important issue in complete markets models with heterogeneous beliefs and time-separable preferences (see e.g. Yan (2008) and Kogan, Ross, Wang, and Westerfield (2006)). Under most models in which agents have identical CRRA preferences, only those agents whose beliefs are closest to the truth will survive in the long run. If the irrational agent (optimist in the foregoing analysis) is quickly eliminated from the economy then the price effects generated by trading between agents disappear. It is therefore worth analyzing the survival of agents A and B, which is defined as their asymptotic share of consumption as the horizon goes to infinity (see e.g. Berrada (2009) and Dumas, Kurshev, and Uppal (2009)).

Table 5 shows the mean and standard deviation of the share of consumption of the optimistic agent, c^B, at horizon $T = 50, 100$, and 500 years; these values are obtained from $1,000$ simulations starting from $c^B = 0.1, 0.5$, and 0.9. I find that the optimist can survive for long periods and that his consumption share actually increases if there are no systemic disasters. Therefore, the risk-sharing dynamics documented previously are not likely to disappear quickly.

The dynamics of the consumption share of the pessimistic agent is obtained by applying Itô's lemma to $c^A = (1 + \phi(t)^{1/\gamma})^{-1}$ and using Equation (4):

$$dc^A(t) = -\frac{1}{\gamma}c^A(t)c^B(t)(\beta^A - \beta^B)X(t)dt + c^A(t)c^B(t)\frac{1 - \left(\frac{\beta^B}{\beta^A}\right)^{1/\gamma}}{c^A(t) + c^B(t)\left(\frac{\beta^B}{\beta^A}\right)^{1/\gamma}}dN_{ct}. \qquad (48)$$

The drift is negative; thus c^A declines deterministically when there is no systemic disaster but increases in response to systemic disaster, and both effects are stronger as the level of disagreement increases.

However, the consumption share's distribution is not stationary, which means that at infinite horizon one of the agents eventually disappears. Such nonstationarity could potentially be an issue in light of an estimation of the model. Possible solutions are provided by Borovicka (2012), who shows that recursive preference specifications lead to equilibria in which both agents survive, and by Garleanu and Panageas (2012), who propose an overlapping-generations framework to obtain a nondegenerate stationary equilibrium. I leave extensions of the model in these directions to future research.

5 Empirical Analysis

This section briefly introduces the data before testing empirically the model's main implications. In particular, I first analyze the link between equity and variance risk premia at an aggregate level via predictive regressions of market excess returns on the variance risk premium. Second, I investigate cross-sectional variations in the forecasting power of the aggregate variance premium.

5.1 Data

The empirical analysis is based on the aggregate $S\&P500$ composite index (a proxy for the aggregate market portfolio) and on returns for each constituent of the $S\&P500$ and for CRSP cap-based portfolios (to analyze cross-sectional implications and the differential effects of small versus big stocks). I use monthly data from January 1990 through December 2011 for a total of 264 monthly observations. Excess returns are constructed by subtracting the log 30-day T-bill yield to the monthly returns, all obtained from CRSP.

The variance risk premium for any asset is defined (see also Section 4.1) as the difference between physical and risk-neutral expectations of total return variance for a given horizon. The Volatility Index (VIX), from the Chicago Board Options Exchange (CBOE) provides a model-free measure of the risk-neutral expectation of total market return variation over the subsequent 30 days and is based on the highly liquid $S\&P500$ index options.[25] The VIX is reported in terms of annualized percentage volatility; however, for consistency with the recent literature on variance risk premia in the stock market,[26] my measure of implied variance (IV) is given by VIX squared and then divided by 12 to obtain a monthly quantity. In order to measure empirically the market variance risk premium (VRP) one also needs a conditional forecast of total return variation under the physical measure. A measure of the realized variance (RV) of the market for a given month can be obtained by summing up $S\&P500$ squared five-minute log returns.[27] Bollerslev, Tauchen, and Zhou (2009) use the average RV over the previous month to approximate the physical expectation of the total return variation. The persistence of volatility renders this approximation a fairly accurate one in general; still, it can produce counterintuitive results in periods of high volatility (i.e., when the persistence of the volatility process is lower; see e.g. Fusari and Gonzalez-Perez (2012)). Thus I compute the expectations under the physical measure of total stock market return variance by a simple projection of the realized variance measure on a set of predictor variables. As in

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[25]See e.g. Carr and Wu (2009) for details on computing the risk-neutral return variation from a portfolio of options. The VIX is subject to some approximation error (see, e.g., the discussion in Jiang and Tian (2007)), but the CBOE procedure for calculating the VIX has emerged as the industry standard. Therefore, relying on the squared VIX as a measure of the risk-neutral expected variance facilitates comparison with other studies.

[26]See Bollerslev, Tauchen, and Zhou (2009), Drechsler (2011), and Drechsler and Yaron (2011).

[27]Several studies suggest that, for highly liquid assets such as the $S\&P500$ index, a five-minute sampling frequency provides a reasonable balance between increasing estimation precision and limiting microstructure noise (see, e.g., the discussion in Hansen and Lunde (2006)). I obtain a monthly time series of realized variance based on five-minute returns from Hao Zhou's webpage: `https://sites.google.com/site/haozhouspersonalhomepage/`.

Drechsler (2011) and Drechsler and Yaron (2011), the realized variance is projected on the value of the squared VIX at the end of the previous month and on a lagged realized variance measure.[28] The difference between the conditional forecast from the projections and the risk-neutral expectation, measured using the VIX, yields the series of one-month market variance premium estimates plotted in Figure 13.[29] The variance premium for the market is negative on average, which means that investors are willing to pay a premium to be insured against high-variance states; the premium is time varying, with periods of a small and smooth premium alternating with periods in which the variance premium is larger (in absolute value) and more volatile. These phases of high and volatile variance premia seem to coincide with periods of large disagreement between investors, denoted by the light gray shaded areas in Figure 13. Periods in which differences in beliefs are large include recessions (denoted by dark gray shaded areas) and other times of financial distress, such as the Long-Term Capital Management crisis and Russian default in 1998. Table 6 gives summary statistics for the estimated variance risk premium and its two constituents: the risk-neutral (implied) and physical expectation of realized variance, denoted respectively by IV and ERV. For the sake of evaluating robustness, Table 6 also shows summary statistics for the period prior to the recent financial crisis (until July 2007).[30] The expected variance measure and the risk premium display significant deviation from normality in both samples, with large skewness and kurtosis. The crisis period is characterized by unprecedented spikes in stock market variance, which is reflected in extremely large standard deviation, skewness, and kurtosis statistics for the ERV full-sample time series; variance risk premium statistics are similar in the two samples. Note also that the variance risk premium is less persistent than the two variance forecasts, with an autocorrelation coefficient of about 0.65.

Variance risk premia for individual stocks can also be computed by using option prices to approximate the risk-neutral expectation of the return variation and using the sum of squared daily returns to approximate realized variance. This procedure is followed by Carr and Wu (2009) for 35 individual stocks and by Buraschi, Trojani, and Vedolin (2013) for the constituents of the $S\&P100$ index. Yet Bollerslev, Tauchen, and Zhou (2009) underscore the importance of using realized variance based on high-frequency data, which are not easily available for single stocks, when estimating the predictive power of variance risk premia for future excess returns.[31] Moreover, there is evidence of a large systematic component

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[28]I also implement the same regressions using an expanding window to rule out any look-ahead bias. Because the results are almost identical, I use the in-sample estimates to facilitate comparison with existing studies and to avoid losing observations at the beginning of the sample for the initial estimation.

[29]Similar dynamics are obtained when using more sophisticated models for the realized variance forecasts, such as the heterogeneous autoregressive (HAR) model of Corsi (2009), but here I focus on the simplest measure because more complex models are difficult to identify using monthly data. Bekaert and Hoerova (2013) compare different volatility forecasting models and show that the projection I use is the best within the simple specifications and also performs relatively well in comparison with more sophisticated models.

[30]The start of the recent financial crisis is usually considered to be August 2007, so my pre-crisis sample ends in July 2007. The results do not change qualitatively if instead I end the pre-crisis period in December 2007 (Jin (2013)), in June 2007 (Drechsler and Yaron (2011)), or even in August 2008—just before the height of the crisis and the two anomalous observations of realized variance for October and November 2008.

[31]An exception is the working paper by Han and Zhou (2011), who use high-frequency stock prices to compute the

in the cross section of variance risk premia (see e.g. Carr and Wu (2009)). Hence I study empirically the forecasting power of the market variance risk premium for both the market excess return and the cross section of stock and portfolio returns. Because the model-implied variance risk premium includes compensation only for jump risk, as a robustness check I run the same predictive regressions using the time series of market variance risk premium due to large jumps as computed by Bollerslev and Todorov (2011), although data are available only for the period 1996–2007.[32] Details on the data and a summary of the results using this alternative measure of the variance risk premium are provided in Appendix B and are generally consistent with the results discussed in this section.

Proxies of belief disagreement are calculated using the mean absolute deviation of one-year-ahead forecasts on real GDP growth from the BlueChip Economic Indicator, which are available at a monthly frequency through December 2009.[33] Being consistent with the model presented in Section 3 would normally require that I measure disagreement about the perceived probability of a systemic disaster, but this is proportional to disagreement about the total expected consumption growth provided agents do agree on the expected growth rate in normal times (see Equation (13)). Another fundamental variable arising from the model is the consumption share of optimistic and pessimistic investors. This consumption share is not observable; however, as proxies for investor optimism the literature has used survey-based measures of investor or consumer sentiment such as the University of Michigan Consumer Sentiment Index (MCSI), the measure of investor sentiment compiled by the American Association of Individual Investors (AAII), and Shiller's Crash Confidence Index (CCI).[34] Some empirical evidence also suggests that these sentiment measures forecast market returns; see for example Charoenrook (2002), Lemmon and Portniaguina (2006), and Edelen, Marcus, and Tehranian (2010).

5.2 Predictive regressions for the market

The simple general equilibrium model in Section 3 implies a tight link between variance risk premia and excess returns of the stock market index, which is consistent with the empirical findings in Bollerslev, Tauchen, and Zhou (2009), Drechsler (2011), and Drechsler and Yaron (2011). Table 7 displays results from ordinary least-squares (OLS) estimation of standard return predictability regressions of the form

$$r_{t+h}^e = \alpha + \beta \, VRP_t + \varepsilon_{t+h}. \tag{49}$$

objective expectation of return variation and then estimate the variance risk premium of a large cross section of stocks.

[32]I thank Viktor Todorov for providing the data.

[33]See Buraschi and Whelan (2011) for details on the database, disagreement measures, seasonal adjustment, and construction of forecasts at fixed one-year horizons. I am grateful to Andrea Vedolin and Paul Whelan for providing the time series of belief disagreement on GDP growth.

[34]The MCSI is one of the most widely followed measures of consumer confidence and has been used extensively in academic research (see e.g. Ludvigson (2004), Lemmon and Portniaguina (2006) and Berkman, Jacobsen, and Lee (2011)). It is available on a monthly basis from January 1978 and is based on surveys conducted for a minimum of 500 households. The AAII asks respondents to classify themselves as bullish, bearish, or neutral on the stock market for the next six months (see e.g. Fisher and Statman (2003)). The CCI refers to the percentage of respondents who state that the probability of a stock market crash occurring within the next two quarters is less than 10% (see e.g. Koulovatianos and Wieland (2011)). Data and explanations can be found on the website http://icf.som.yale.edu/stock-market-confidence-indices-united-states.

I regress monthly $S\&P500$ excess returns—at horizons h ranging from one month to one year—on the variance risk premium. The excess return series for $h > 1$ are overlapping, t-statistics are Newey–West corrected, and I report adjusted R^2 in percentage. Panel A reports regression estimates for the full sample of 264 monthly observations, while Panel B is limited to the pre-crisis sample. In line with results reported in the literature, there is a negative and significant relation between the variance premium and excess returns; also, the predictive power (measured either as the adjusted R^2 or as t-statistics of the regression coefficient) is highest between the three-month and the six-month horizon. In particular, for the pre-crisis sample the R^2 peaks at a quarterly horizon and the regression coefficients become insignificant at long horizons; these findings are consistent with those of Bollerslev, Tauchen, and Zhou (2009), whose sample ends in December 2007. Including the financial crisis (Panel A) yields stronger results and a variance risk premium that significantly predicts excess returns also at longer horizons, consistently with Fusari and Gonzalez-Perez (2012). Panels C and D report results from robust regressions that employ Huber-type weights to limit the influence of outliers. The robust regression estimates agree both in magnitude and sign with the OLS estimates, and in most cases the predictability evidence is even stronger.[35] Overall, these results indicate a considerable ability of the variance risk premium to predict future market excess returns.

Stability analysis and flexible regression methods

Estimating the regression in Equation (49) implicitly imposes major restrictions on the relation between variance risk premium and future returns, since that regression assumes a monotone and linear structure. The theoretical asset pricing model presented in Section 3 suggests that this relation is only conditionally linear; unconditionally the relation need not be linear or even monotonic. Therefore, I study empirically potential instabilities or nonlinearities in the standard regression results. Introducing additional regressors does not qualitatively change the results (as shown by Bollerslev, Tauchen, and Zhou (2009)), so I rely on simple regressions to outline the properties of the relation between returns and variance premia. I also focus on the six-month horizon, for which significance of the standard predictive regressions seems to be stronger. The regression coefficient β for a large market index should vary with the distribution of the consumption share between agents (see Section 4.4). That distribution is not observable, but in the model it is directly linked to the level and volatility of the variance risk premium (see Table 2); hence I investigate the shape of the predictive relation for different levels of the premia.

First, I run regression (49) separately for different quantiles of the variance risk premium. Figure 14 plots the distributions of regression coefficients (upper panel) and R^2 (lower panel), which are obtained by applying a block bootstrap procedure. In both panels, the leftmost box plot corresponds to small absolute values of the premium ($VRP < q_{70\%}$), the rightmost one to large values ($VRP > q_{30\%}$), and

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[35]A more naive way to control for the effect of outliers is to run the OLS regression without the two potentially anomalous observations of October and November 2008 (at the peak of the financial crisis) when the realized variance experienced unprecedented levels. Results do not qualitatively change, and the estimates are just slightly more significant than in Panel A of Table 7.

the middle box plot to average values of the VRP. In accordance with the model, predictive power is increasing in the (absolute) level of the variance premium and the regression coefficient is significantly different from zero only for large values of the variance risk premium. From an empirical standpoint, changes in the variance premium are of course not exclusively related to changes in the cross-sectional consumption distribution of disagreeing agents. In order to relate more tightly the instability of standard predictive regressions to the extent of risk sharing among agents, I stratify regression (49) according to the level of difference in beliefs (DB). Figure 15 shows the distributions of regression coefficients (upper panel) and R^2 (lower panel) for small, average, and large DB values (once again a block bootstrap procedure is used). The regression coefficient increases in absolute value with the level of DB, from -0.2415 to -0.9725; the adjusted R^2 is 1.09% for small DB and increases to 4.02% and 20.91% for average and large DB, respectively.[36] The link between the level of the variance premium and measures of disagreement is confirmed by a simple OLS regression of monthly VRP on DB from January 1990 through December 2009. A change of one standard deviation in DB yields a change of 0.38 standard deviations in the variance premium; this result is strongly significant both statistically and economically, with a Newey–West corrected t-statistic of -4.319 and an adjusted R^2 of about 14%.[37]

A second way to analyze the validity of a simple linear regression is to estimate a fully nonparametric regression of the form

$$r^e_{t+h} = m(VRP_t) + \varepsilon_{t+h}, \tag{50}$$

where $m \colon \mathbb{R} \to \mathbb{R}$ is an arbitrary function fulfilling some smoothness conditions. An estimate of the function m can be simply obtained by using the Nadaraya–Watson kernel estimator with, in this univariate case, a Gaussian kernel; see Figure 16.[38] The number of observations is not large enough to draw strong conclusions, but a visual inspection clearly confirms the absence of any link between the two variables for small (absolute) values of the variance risk premia, although there is a stronger negative relation for more extreme values of those premia. Hence this simple nonparametric analysis supports the conclusion that the predictive power of variance risk premia for market returns is a time-varying and nonlinear phenomenon.

To avoid a fully nonparametric procedure, it is possible to model explicitly the regression coefficient's time variation. The conditional β could, for example, be a function of the variance premium itself:

$$\tilde{r}^e_{t+h} = \underbrace{(\beta_0 + \beta_1 \widetilde{VRP_t})}_{\beta_t} \widetilde{VRP_t} + \varepsilon_{t+h}, \tag{51}$$

where the tilde marks variables that are standardized. Equation 51 is equivalent to a quadratic regression

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[36]The values of the regression coefficients and R^2 are not exactly comparable to the results obtained previously because DB is available only until December 2009 (see Section 5.1).

[37]The level of the variance premium is positively linked also to measures of sentiment, such as the as the MCSI, with a correlation of almost 20% between the two measures. However, the results are less significant, probably because sentiment measures—which are based on surveys at the household level—are relatively noisy.

[38]The optimal bandwidth, computed as suggested by Bowman and Azzalini (1997), is 4.77.

and can be estimated via standard OLS.[39] If β_1 proved to be insignificant then we could not reject a linear relation between excess returns and lagged variance premia, but β_1 is actually both positive and significant whereas β_0 (the linear term) is insignificant; the adjusted R^2 of this regression is 9.47%, which corresponds to a 22% increase over the linear regression's adjusted R^2 of 7.75% (at six-month horizon). In general, time variation in the regression coefficient is modeled by introducing an interaction term. Apart from the level of VRP, other reasonable candidates worth exploring are the volatility of VRP and the level of disagreement or optimism. If we use DB as a conditioning variable, then the regression coefficient β_t becomes more negative with increasing disagreement (as expected), and the adjusted R^2 for the monthly 1990–2009 sample increases from 6.3% to 7.6%.

An alternative way of analyzing the time variation in the predictive power of the variance premium for future returns is to estimate a standard regression on a rolling window. Toward that end, I regress market excess returns at a 6-month horizon on lagged variance risk premium using a rolling window of 50 months. Figure 17 reports estimates of the slope coefficient and corresponding adjusted R^2. Instability of the predictive relation is evident, and it is possible to relate the time variation in the slope coefficient to measures of disagreement. The correlation between the rolling regression coefficient estimate and a moving average of the difference in belief measure is equal to -51.18%, which means that the regression coefficient becomes more negative as disagreement increases. The relation between the regression's slope coefficient and the difference in beliefs suggests that the predictive power of variance risk premia for future excess returns is countercyclical, given that measures of disagreement are known to increase in bad times (see e.g. Patton and Timmermann (2010) and Buraschi, Trojani, and Vedolin (2011)).

A growing body of empirical evidence documents instabilities and nonlinearities in the strength of the return predictability by popular macroeconomic variables such as the dividend yield and short rate variables. For example, Henkel, Martin, and Nardari (2011) use a regime-switching model to show that standard aggregate return predictors are effective during business cycle contractions but practically useless during expansions. In the same way, I examine the dynamics in the predictive power of variance risk premia via estimation of a regime-switching model:

$$\tilde{r}^e_{t+h} = \beta_s \, \widetilde{VRP}_t + \varepsilon_{s,t+1}, \tag{52}$$

where $\varepsilon_s \sim N(0, \sigma_s^2)$ and the state $s \in \{1, 2\}$ follows a Markov chain with constant transition probabilities. I find that predictability is present only in state 2, which is characterized by more volatile and larger (on average, in absolute value) variance risk premia (see Figure 18). In state 1, the regression coefficient is positive and insignificant. The estimated transition probability matrix is

$$\begin{bmatrix} 0.97 & 0.03 \\ 0.07 & 0.93 \end{bmatrix};$$

thus states are persistent and with expected durations of about 3.27 and 1.24 years, respectively. It is most interesting that state 2 corresponds to periods of financial crisis. In particular: the first shaded

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[39] In other words, I am estimating Equation (50) while requiring that m be a quadratic function. One could, theoretically, employ other functional forms, but this is the most obvious alternative to a linear regression.

area in Figure 18 corresponds to the US savings and loans crisis; the second, starting in June 1996, includes the Asian financial crisis, the Russian default, and the bursting of the dot-com bubble; and the last shaded area starts in September 2007 with the recent financial crisis. Therefore, the shift in regime of the variance premium (and of its predictive power) could be linked to a systemic disaster or to a jump in the consumption share of pessimistic agents, as would be implied by the simple model in Section 3.

5.3 Predictive regressions in the cross section

I next test the hypothesis that market variance risk premia predict excess returns also for single stocks and portfolios, consistently with the model in Section 3 and with the empirical evidence that aggregate variance premium is a priced factor (see e.g. Carr and Wu (2009)). If investors are averse to variance risk, then stocks with high (negative) predictive VRP loadings will have higher expected returns and stocks with low or positive regression coefficients will serve as hedges and thus have lower expected returns. In line with the foregoing aggregate predictive analysis, I estimate regressions of the form

$$r_{i,t+h}^e = \alpha_i + \beta_i \, VRP_t + \varepsilon_{t+h}, \tag{53}$$

where $r_{i,t+h}^e$ denotes the monthly excess returns on stock i with horizon h. Average regression coefficients are similar to those obtained for the stock index in Panel A of Table 7, but on average they are not significant owing to a large cross-sectional variation. So as to understand better the cross-sectional dynamics, I sort stocks based on their estimated variance risk premium loading, β_i, at horizon $h = 6$ months. Table 8 reports average estimation results in each quintile of β-values. In the first quintile (Panel A), the regression coefficient at any horizon is about 3 times larger than for the market index and the adjusted R^2 is higher at horizons of 6, 9, and 12 months. In the second quintile (Panel B), the values of the regression coefficient are close to those of the market but are significant only at horizons of 3, 6, and 9 months; in general, significance and R^2 decline rapidly as the value of beta increases (Panels C, D, and E). The regression coefficients switch sign in the fifth quintile but are not significantly different from zero. There seems to be a link between the value of the variance risk premium loading and stock capitalization, which is consistent with the model (see Section 4.3). In fact, the average capitalization of stocks in the first quintile is about 40% less than that of stocks in the fifth quintile. To investigate this link further, I run predictive regression (53) using CRSP cap-based portfolio returns and report the estimation results in Table 9. Decile 1 includes the largest stocks and decile 10 the smallest; results are reported for portfolios including deciles 1 and 2 (large-cap CRSP index), 3 to 5 (mid-cap CRSP index), and 6 to 10 (small-cap CRSP index). At the one-month horizon, there does not seem to be a clear pattern and overall significance is quite weak. At longer horizons, the predictive power of the market variance risk premium for excess returns is much stronger for small stocks, in line with the model and with the empirical evidence described previously. At the six-month horizon, for instance, adjusted R^2 for the small-cap portfolio is 10.17%—more than 50% larger than the 6.21% R^2 for the big-cap portfolio. The forecasting power with respect to small stocks is still impressive at the one-year horizon, with an R^2 of

9.67%. As a robustness check, I estimate the same regression on the 25 Fama and French portfolios that are sorted by the firm characteristics of size and book-to-market ratio (BM).[40] Figure 19 plots estimates of VRP loadings (left panel) and of adjusted R^2 in percentage (right panel) from regression (53), at the six-month horizon, for the 25 Fama and French portfolios. Lines connect portfolios of different book-to-market categories within each size category while focusing on the bottom and upper quintiles, which correspond to small and big stocks, respectively. On average, small stocks have larger (in absolute value) VRP loading and higher R^2. This means that exposure to aggregate variance risk could partially explain the size premium.[41] The predictive power of variance risk premium for future returns seems to be stronger also for growth stocks. Within the simulated model, however, no distinction can be drawn between size and value effects if assets have identically distributed cash flows.[42]

As in the case of predictive regressions for the aggregate market, discussed in Section 5.2, the regression model in Equation (53) likewise assumes a monotone and linear structure (this is contrary to implications of the theoretical asset pricing model presented in Section 3). Therefore, also in the cross section I analyze the dependence of the linear regression coefficient β_i on the level of the variance risk premium and of the difference in beliefs (focusing on the six-month horizon); results are consistent with those reported in Section 5.2. Figure 20 shows the results of estimating regression (53), using CRSP cap-based portfolio returns, separately for different quantiles of the difference in beliefs (DB). Left panels display the distributions of regression coefficients and right panels display adjusted R^2, where both are obtained by applying a block bootstrap procedure. Within each panel, the leftmost box plot corresponds to small values of the disagreement ($DB < q_{30\%}$), the rightmost to large values ($DB > q_{70\%}$), and the middle box plot to average values of DB. In accordance with the model and just as for the aggregate predictive regression (Figure 14), here predictive power increases with the level of the difference in beliefs. Table 10 summarizes these results. Differences among the DB quantiles seem to be stronger for the small-cap portfolio. Also, the difference between small- and big-cap beta is significant only in the state where there is a large difference in beliefs. In other states, the VRP loading and the adjusted R^2 are strongly similar for large and small stocks, and the big-cap portfolio beta is even higher (in absolute value) than the beta for small stocks when disagreement is low. The last panel of Table 10 reports results of regressing, on the aggregate variance risk premium, the return of a portfolio that is long the small-cap index and short the big-cap index. The regression coefficient is negative and strongly significant when disagreement is high, with an R^2 of more than 20%. Therefore, in these states a large (absolute) variance risk premium predicts a larger size premium, while the effect is not significant when difference in beliefs

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[40]Data are obtained from Kenneth French's website. The portfolios are the intersections of five portfolios formed on size and five portfolios formed on the ratio of book equity to market equity. Breakpoints are the NYSE market equity and *BM* quintiles. *Size 1* corresponds to small stocks and *BM 1* to growth stocks.

[41]Bali and Zhou (2011) show that an asset pricing model in which both market risk and aggregate variance risk premium are priced can explain the premia for industry, size, and value.

[42]Martin (2013) proposes different ways to generate variation along the value dimension that does not align perfectly with size in a general equilibrium model with multiple trees. He shows that the different alternatives generate very different patterns of alphas and betas across the size and value dimensions.

is low. This finding is related to the work of Lemmon and Portniaguina (2006), who show a negative relation between the size premium and consumer confidence. In fact, the size effect (of smaller firms having higher returns on average) seems to be concentrated in periods characterized by large disagreement, as shown in Table 11. It is natural for investors to require higher returns on assets that are more sensitive to systemic disaster risk. The economic intuition for this finding can be found in Jagannathan and Wang (1996), who argue that firms more likely to exhibit financial distress (e.g., small firms) have market betas that are more sensitive to changes in the business cycle. Investor sentiment, or disagreement, is thus related to time variation in the expected returns of those firms because these factors forecast future business conditions. However, this reasoning holds only when the perceived systemic jump premium is high. The model suggests that the systemic jump premium component could actually have a negative effect on the stock's excess returns if the consumption share of pessimists were low enough. Thus the size premium could go in opposite directions depending on which agent type dominates the market. This finding is consistent with some of the later empirical research on the size effect, which suggests that the premium disappears in the 1980s.

6 Conclusion

This paper studies both theoretically and empirically how agent disagreement about the likelihood of systemic disasters affects the equity and variance risk premia and the relation between them, both for the market portfolio and in the cross section of stocks. The starting point is a general equilibrium model with multiple trees and disagreement about systemic rare event risk.

The main findings are the following. First, the equity (variance) risk premium of an individual stock tends to increase (decrease) with its dividend share and with the consumption share of the pessimistic agent. The variance risk premium can also switch sign, mainly for small stocks, and it is time varying; it alternates phases of small and smooth premia with periods in which the variance premium is larger (in absolute value) and more volatile. Second, the index variance risk premium is largely due to a covariance premium when assets are relatively evenly distributed or the number of stocks in the economy is large. The model-implied correlation risk premium, as the variance risk premium, is large (in absolute value) when pessimists hold a large fraction of the aggregate endowment. Third, rare event risk implies a tight link between the equity and the variance risk premia, both for the market and for the cross section of stock returns. This link however is state-dependent and varies with the asset's dividend share and the agents' consumption share. In particular, the relation is stronger when the consumption share of the pessimist is larger, i.e., in bad states of the economy, and for small stocks. Fourth, in the case of a large diversified economy only systemic risk is priced and the relation between equity and variance risk premia is conditionally linear. Moreover, infinitely small assets still earn a risk premium owing to the presence of systemic rare event risk.

I test empirically the main predictions and show that, as implied by the model, the relation between

the equity premium and the index variance premium is time varying and systematically linked to the degree of risk sharing among disagreeing investors. In particular, the predictive power of variance premium for future excess returns is stronger in periods of large differences in investor beliefs. This relation holds especially for small stocks, whose returns are more dependent on the compensation for systemic rare event risk.

This work suggests several interesting lines of future research. On the theoretical side, the model's simplicity means that several extensions are possible. Examples include introducing a stochastic diffusion volatility of the dividend processes and allowing for learning based on an exogenous signal about the state of the economy. I could also introduce disagreement with regard to both the disaster intensity and the expected dividend growth in normal times. Given the link between volatility risk premia and option surfaces, the option pricing implications of an heterogeneous rare disaster model are also worth exploring.

On the empirical side, it would be natural to look for potential time variation and nonlinearity in the relation between stock returns and correlation risk premia as I find for the variance risk premium. It would also be worth investigating whether the same nonlinear relationships are present in other markets for which a link between disagreement or variance premia and excess returns has been documented, as for example the fixed income market (Mueller, Vedolin, and Yen (2011) and Buraschi and Whelan (2011)) and the foreign exchange market (Beber, Breedon, and Buraschi (2010)).

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A Proofs

A.1 Proof of Proposition 1

Given multiplier ξ for the constraint in representative agent problem (17),

$$U^A(C^A(t))' = \xi = \phi(t)U^B(C^B(t))' \quad \Rightarrow \quad \phi(t) = \frac{U^A(C^A(t))'}{U^B(C^B(t))'} = \frac{y^A\eta^A(t)}{y^B\eta^B(t)}, \tag{54}$$

where the last equality follows from individual agents optimality. Equation (54) and the optimality condition of agent A imply:

$$U(C(t),\phi(t))' = U^A(C^A(t))'\frac{\partial C^A}{C} + \phi U^B(C^B(t))'\frac{\partial C^A}{C} = U^A(C^A(t))' = y^A e^{\delta t}\eta^A(t). \tag{55}$$

Therefore, individual agents optimal consumptions can be written as

$$C^A(t) \;=\; (y^A e^{\delta t}\eta^A(t))^{-1/\gamma} = U'(C(t),\phi(t))^{-1/\gamma}, \tag{56}$$

$$C^B(t) \;=\; (y^B e^{\delta t}\eta^B(t))^{-1/\gamma} = \left(\frac{y^A e^{\delta t}\eta^A(t)}{\phi(t)}\right)^{-1/\gamma} = \left(\frac{U'(C(t),\phi(t))}{\phi(t)}\right)^{-1/\gamma}. \tag{57}$$

Using the market clearing condition $C(t) = C^A(t) + C^B(t)$,

$$C(t) = U'(C(t),\phi(t))^{-1/\gamma} + \left(\frac{U'(C(t),\phi(t))}{\phi(t)}\right)^{-1/\gamma},$$

which can be solved for the marginal utility of the representative agent, leading to

$$U'(C(t),\phi(t)) = \frac{(1+\phi(t)^{1/\gamma})^\gamma}{C(t)^\gamma}. \tag{58}$$

Inserting (58) in (56) and (57) lead to the equilibrium consumption allocations in equation (19) and the investors' state price densities (20).

A.2 Proof of Proposition 2

The state price density of agent A is given in Proposition 1:

$$\eta^A(t) = e^{-\delta t}\frac{(1+\phi(t)^{1/\gamma})^\gamma}{y^A C(t)^\gamma}$$

Applying Ito's lemma and using the dynamics of $\phi(t)$ and $C(t)$ in equations (4) and (11), respectively, the dynamics of η^A is given by:

$$\begin{aligned}
d\eta^A(t) \;=\; & -\delta\eta^A(t)dt + \eta^A(t)\frac{\phi(t)^{1/\gamma}}{1+\phi(t)^{1/\gamma}}(\beta^A-\beta^B)X(t)dt - \gamma\eta^A(t)(\mu dt + \sigma\sum_{j=1}^N s_j dW_{jt}) + \\
& +\frac{1}{2}\gamma(\gamma+1)\eta^A(t)\sigma^2\sum_{j=1}^N s_j^2 dt + \eta^A(t)\sum_{j=1}^N\left[(s_j k+1)^{-\gamma}-1\right]dN_{jt} + \\
& +\eta^A(t)\left[\left(\frac{1+\left(\phi(t)\frac{\beta^B}{\beta^A}\right)^{1/\gamma}}{1+\phi(t)^{1/\gamma}}\right)^\gamma(k+1)^{-\gamma}-1\right]dN_{ct}.
\end{aligned} \tag{59}$$

Therefore,

$$
\begin{aligned}
\frac{d\eta^A(t)}{\eta^A(t)} &= \left[-\delta + \frac{\phi(t)^{1/\gamma}}{1+\phi(t)^{1/\gamma}}(\beta^A - \beta^B)X(t) - \gamma\mu + \frac{1}{2}\gamma(\gamma+1)\sigma^2 \sum_{j=1}^{N} s_j^2 \right] dt - \gamma\sigma \sum_{j=1}^{N} s_j dW_{jt} + \\
&\quad + \sum_{j=1}^{N} \left[(s_j k + 1)^{-\gamma} - 1 \right] dN_{jt} + \left[\left(\frac{1 + \left(\phi(t)\frac{\beta^B}{\beta^A} \right)^{1/\gamma}}{1 + \phi(t)^{1/\gamma}} \right)^{\gamma} (k+1)^{-\gamma} - 1 \right] dN_{ct}. \qquad (60)
\end{aligned}
$$

Comparing the drift, diffusion and jump terms of this expression with those in equation (16) directly leads to the solution for θ_j, λ_j^Q, λ_c^Q and r in Proposition 2.

A.3 Proof of Proposition 3

For simplicity, I first provide the derivation of the stock price expression for the case of a two-trees economy, to then extend it to the case of N stocks.

A.3.1 Stock prices in an economy with $N = 2$ stocks

For notational convenience, I compute the price of stock 1, the price of stock 2 follows an analogous expression with obvious modifications. Let me define $y_{it} \equiv \ln D_i(t)$ and $\tilde{y}_{i\tau} \equiv y_{i\tau} - y_{it}$. The price of stock 1 is given by the discounted value of all its future dividends:

$$
S_1(t) = E_t^A \left[\int_t^T \frac{\eta^A(s)}{\eta^A(t)} D_1(s) ds \right]. \qquad (61)
$$

This can also be viewed as a portfolio of zero coupon dividend claims:

$$
S_1(t) = \int_t^T S_1^\tau(t) d\tau
$$

with

$$
\begin{aligned}
S_1^\tau(t) &= E_t^A \left[\frac{\eta^A(\tau)}{\eta^A(t)} D_1(\tau) \right] \\
&= e^{-\delta(\tau-t)} \frac{C(t)^\gamma}{(1+\phi(t)^{1/\gamma})^\gamma} E_t^A \left[\frac{(1+\phi(\tau)^{1/\gamma})^\gamma}{C(\tau)^\gamma} D_1(\tau) \right] \\
&= e^{-\delta(\tau-t)} \frac{C(t)^\gamma}{(1+\phi(t)^{1/\gamma})^\gamma} E_t^A \left[\sum_{k=0}^{\gamma} \binom{\gamma}{k} \phi(\tau)^{k/\gamma} \frac{e^{y_{1t}+\tilde{y}_{1\tau}}}{(e^{y_{1t}+\tilde{y}_{1\tau}} + e^{y_{2t}+\tilde{y}_{2\tau}})^\gamma} \right] \\
&= e^{-\delta(\tau-t)} \frac{C(t)^\gamma}{(1+\phi(t)^{1/\gamma})^\gamma} E_t^A \left[\sum_{k=0}^{\gamma} \binom{\gamma}{k} \phi(\tau)^{k/\gamma} \frac{e^{(1-\gamma/2)y_{1t}-\gamma/2 y_{2t}+(1-\gamma/2)\tilde{y}_{1\tau}-\gamma/2\tilde{y}_{1\tau}}}{\left(2\cosh\left(\frac{y_{2t}-y_{1t}+\tilde{y}_{1\tau}-\tilde{y}_{2\tau}}{2} \right) \right)^\gamma} \right] \\
&= e^{-\delta(\tau-t)} \frac{C(t)^\gamma}{(1+\phi(t)^{1/\gamma})^\gamma} e^{(1-\gamma/2)y_{1t}-\gamma/2 y_{2t}} \sum_{k=0}^{\gamma} \binom{\gamma}{k} E_t^A \left[\frac{e^{k/\gamma \ln\phi(\tau)+(1-\gamma/2)\tilde{y}_{1\tau}-\gamma/2\tilde{y}_{1\tau}}}{\left(2\cosh\left(\frac{y_{2t}-y_{1t}+\tilde{y}_{1\tau}-\tilde{y}_{2\tau}}{2} \right) \right)^\gamma} \right] \quad (62)
\end{aligned}
$$

assuming an integer coefficient of relative risk aversion γ. Then I use the fact that

$$
\frac{1}{[2\cosh(u/2)]^\gamma} = \int_{-\infty}^{\infty} e^{iuz} \mathcal{F}_\gamma(z) dz,
$$

where the Fourier transform $\mathcal{F}_\gamma(z)$ is given by (see Martin (2013)):

$$\mathcal{F}_\gamma(z) \equiv \frac{\Gamma(\gamma/2 + iz)\Gamma(\gamma/2 - iz)}{2\pi\Gamma(\gamma)}. \tag{63}$$

The conditional expectation in Equation (62) can thus be written as

$$
\begin{aligned}
E_t^A\left[\frac{e^{k/\gamma \ln \phi(\tau) + (1-\gamma/2)\tilde{y}_{1\tau} - \gamma/2\tilde{y}_{1\tau}}}{\left(2\cosh\left(\frac{y_{2t} - y_{1t} + \tilde{y}_{1\tau} - \tilde{y}_{2\tau}}{2}\right)\right)^\gamma}\right] &= \int_{-\infty}^\infty \mathcal{F}_\gamma(z)e^{iz(y_{2t}-y_{1t})}E_t^A\left[e^{k/\gamma \ln \phi(\tau) + (1-\gamma/2-iz)\tilde{y}_{1\tau} + (-\gamma/2+iz)\tilde{y}_{2tau}}\right]dz \\
&= \int_{-\infty}^\infty \mathcal{F}_\gamma(z)e^{iz(y_{2t}-y_{1t})}D_1(t)^{-(1-\gamma/2-iz)}D_2(t)^{-(iz-\gamma/2)} \\
&\quad \cdot e^{(\tau-t)[(1-\gamma)(\mu-1/2\sigma^2)+1/2\sigma^2((1-\gamma/2-iz)^2+(-\gamma/2-iz)^2)+(1-\gamma/2-iz)y_1^c+(-\gamma/2-iz)y_2^c]} \\
&\quad \cdot E_t^A\left[e^{k/\gamma \ln \phi(\tau) + (1-\gamma/2-iz)y_1^d(\tau) + (-\gamma/2+iz)y_1^d(\tau)}\right]dz, \tag{64}
\end{aligned}
$$

where $y_i^c(t)$ and $y_i^d(t)$, for $i = 1,2$ are the diffusion and jump components of log dividends and their dynamics are given by:

$$dy_i^c(t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_{it}, \tag{65}$$

$$dy_i^d(t) = \ln(k+1)(dN_{it} + dN_{ct}), \tag{66}$$

and

$$d\ln \phi(t) = (\beta^A - \beta^B)X(t)dt + \ln\left(\frac{\beta^B}{\beta^A}\right)dN_{ct}, \tag{67}$$

from equation (4).

Thus, denoting $x(t) \equiv \frac{k}{\gamma}\ln \phi(\tau) + (1 - \gamma/2 - iz)y_1^d(\tau) + (-\gamma/2 + iz)y_1^d(\tau)$, its dynamics follows:

$$
\begin{aligned}
dx(t) = {}& \frac{k}{\gamma}(\beta^A - \beta^B)X(t)dt + \underbrace{(1-\gamma/2-iz)\ln(k+1)}_{k_x^1}dN_{1t} + \underbrace{(iz-\gamma/2)\ln(k+1)}_{k_x^2}dN_{2t} \\
& + \underbrace{\left[(1-\gamma)\ln(k+1) + \frac{k}{\gamma}\ln\left(\frac{\beta^B}{\beta^A}\right)\right]}_{k_x^c}dN_{ct}, \tag{68}
\end{aligned}
$$

and also depends on the state variable $X(t)$. Let me define $Y = \begin{bmatrix} x \\ X \end{bmatrix}$, whose dynamics can be written as

$$dY = \left(\begin{bmatrix} 0 \\ \varphi \end{bmatrix} + \begin{bmatrix} 0 & \frac{k}{\gamma}(\beta^A - \beta^B) \\ 0 & -\varphi \end{bmatrix}Y\right)dt + \begin{bmatrix} 0 \\ \sigma_X\sqrt{X} \end{bmatrix}dW_t^X + \begin{bmatrix} k_x^1 \\ 0 \end{bmatrix}dN_{1t} + \begin{bmatrix} k_x^2 \\ 0 \end{bmatrix}dN_{2t} + \begin{bmatrix} k_x^c \\ 0 \end{bmatrix}dN_{ct}, \tag{69}$$

with jump intensities $\lambda_Y^1 = \lambda_Y^2 = \lambda$ and $\lambda_Y^c = [0 \quad \beta^A]Y$.

The conditional expectation in equation (64) can thus be written as

$$f(Y,t) = E_t^A\left[e^{wY}\right],$$

with $w = [1 \quad 0]$ and since Y follows an affine jump diffusion we know (see Duffie, Pan, and Singleton (2000)) that $f(Y,t)$ is of the form $f(Y,t) = e^{\alpha_{0,k}(s) + \alpha_{1,k}(s)x(t) + \alpha_{2,k}(s)X(t)}$, where $s = \tau - t$ and

$\alpha_{0,k}(s), \alpha_{1,k}(s)$ and $\alpha_{2,k}(s)$ follow the system of Riccati equations:

$$\alpha'_{0,k}(s) = \alpha_{2,k}(s)\varphi + \lambda\left(e^{k_x^1} - 1 + e^{k_x^2} - 1\right), \tag{70}$$

$$\alpha'_{1,k}(s) = 0, \tag{71}$$

$$\alpha'_{2,k}(s) = \frac{k}{\gamma}(\beta^A - \beta^B)\alpha_{1,k}(s) - \varphi\alpha_{2,k}(s) + \frac{1}{2}\sigma_X^2\alpha_{2,k}(s)^2 + \beta^A\left(e^{k_x^c} - 1\right), \tag{72}$$

with initial conditions, $\alpha_{0,k}(0) = \alpha_{2,k}(0) = 0$ and $\alpha_{1,k}(0) = 1$.

From (71) we find $\alpha_{1,k}(s) = \alpha_{1,k}(0) = 1$, while $\alpha_{0,k}(s)$ and $\alpha_{2,k}(0)$ are easily solved numerically.[43]

Therefore, $f(Y,t) = e^{\alpha_{0,k}(s)+x(t)+\alpha_{2,k}(s)X(t)}$, and the price of a zero coupon dividend claim on the is

$$
\begin{aligned}
S_1^\tau(t) &= e^{-\delta(\tau-t)}\frac{C(t)^\gamma}{(1+\phi(t)^{1/\gamma})^\gamma}\frac{D_1(t)}{(D_1(t)D_2(t))^{\gamma/2}}\sum_{k=0}^{\gamma}\binom{\gamma}{k}\phi(t)^{k/\gamma}\int_{-\infty}^{\infty}\mathcal{F}_\gamma(z)e^{iz(y_{2t}-y_{1t})}\\
&\quad \cdot e^{(\tau-t)[(1-\gamma)(\mu-1/2\sigma^2)+1/2\sigma^2((1-\gamma/2-iz)^2+(-\gamma/2-iz)^2)]+\alpha_{0,k}(\tau-t)+\alpha_{2,k}(\tau-t)X(t)}dz\\
&= D_1(t)\left(\frac{C(t)}{\sqrt{D_1(t)D_2(t)}}\right)^\gamma\sum_{k=0}^{\gamma}\binom{\gamma}{k}\frac{\phi(t)^{k/\gamma}}{(1+\phi(t)^{1/\gamma})^\gamma}\int_{-\infty}^{\infty}\mathcal{F}_\gamma(z)e^{izu_t}\\
&\quad \cdot e^{(\tau-t)[-\delta+(1-\gamma)(\mu-1/2\sigma^2)+1/2\sigma^2((1-\gamma/2-iz)^2+(-\gamma/2-iz)^2)]+\alpha_{0,k}(\tau-t)+\alpha_{2,k}(\tau-t)X(t)}dz \tag{73}
\end{aligned}
$$

Therefore the price of the aggregate consumption claim is

$$
\begin{aligned}
S_1(t) &= \int_t^T S_1^\tau(t)d\tau\\
&= D_1(t)\left[2\cosh\left(\frac{u_t}{2}\right)\right]^\gamma\sum_{k=0}^{\gamma}a_k(\phi)\int_{-\infty}^{\infty}\mathcal{F}_\gamma(z)e^{izu_t}b_k(X,t,z)dz, \tag{74}
\end{aligned}
$$

where

$$a_k(\phi) = \binom{\gamma}{k}\frac{\phi(t)^{k/\gamma}}{(1+\phi(t)^{1/\gamma})^\gamma}, \tag{75}$$

$$b_k(X,t) = \int_t^T e^{(\tau-t)[-\delta+(1-\gamma)(\mu-1/2\sigma^2)+1/2\sigma^2((1-\gamma/2-iz)^2+(-\gamma/2-iz)^2)]+\alpha_{0,k}(\tau-t)+\alpha_{2,k}(\tau-t)X(t)}d\tau \tag{76}$$

and the price-dividend ratio of stock 1 is

$$g_1(\phi,X,u,t) \equiv \frac{S_1(t)}{D_1(t)} = \left[2\cosh\left(\frac{u_t}{2}\right)\right]^\gamma\sum_{k=0}^{\gamma}a_k(\phi)\int_{-\infty}^{\infty}\mathcal{F}_\gamma(z)e^{izu_t}b_k(X,t,z)dz.$$

In the same way it is possible to obtain the closed form expression for the price of stock 2, $S_2(t)$.

A.3.2 General stock price expressions in an economy with N stocks

The basic approach is the same with $N > 2$ assets. The main technical difficulty lies in generalizing $\mathcal{F}_\gamma(z)$ to the N-asset case, but this problem is solved by Martin (2013), who defines

$$\mathcal{F}_\gamma^N(\mathbf{z}) \equiv \frac{\Gamma(\gamma/N + iz_1 + iz_2 + \ldots + iz_{N-1})}{(2\pi)^{N-1}\Gamma(\gamma)}\prod_{k=1}^{N-1}\Gamma(\gamma/N - iz_k). \tag{77}$$

The price-dividend ratio on an asset j is thus:

$$g_j(\phi,X,\mathbf{u},t) = e^{-\gamma\sum_{j=2}^N u_j/N}(1+e^{u_2}+\ldots+e^{u_N})^\gamma\sum_{k=0}^{\gamma}a_k(\phi)\int\mathcal{F}_\gamma^N(\mathbf{z})e^{i\mathbf{u}'\mathbf{z}}b_{jk}(X,t,\mathbf{z})d\mathbf{z}, \tag{78}$$

---

[43]In my simulations I use a Runge-Kutta 4th order method.

where the integral is evaluated on $\mathbb{R}^{N-1}$, $a_k(\phi)$ is given in Equation (75) and $b_{jk}(X, t, \mathbf{z})$ generalizes Equation (76) as follows:

$$b_{jk}(X, t, \mathbf{z}) = \int_t^T e^{(\tau-t)\left[-\delta+(\mu-\frac{1}{2}\sigma^2)\mathbf{1}_N'(\mathbf{e}_j-\gamma/N+i\mathbf{U}'\mathbf{z})+\frac{1}{2}\sigma^2(\mathbf{e}_j-\gamma/N+i\mathbf{U}'\mathbf{z})'(\mathbf{e}_j-\gamma/N+i\mathbf{U}'\mathbf{z})\right]+\alpha_{0,k}^N(\tau-t)+\alpha_{2,k}^N(\tau-t)X(t)} d\tau,$$

where $\mathbf{e}_j$ is the N-vector with a 1 at the j-th entry and zeros elsewhere, $\mathbf{1}_N$ is a N-dimensional vector of ones and $\alpha_{0,k}^N(\tau)$ and $\alpha_{2,k}^N(\tau)$ satisfy the following system of Riccati equations:

$$\alpha_{0,k}^N(s)' = \alpha_{2,k}^N(s)\varphi + \lambda \sum_{i=1}^N \left(e^{k_x^i} - 1 \right),$$

$$\alpha_{2,k}^N(s)' = \frac{k}{\gamma}(\beta^A - \beta^B) - \varphi \alpha_{2,k}^N(s) + \frac{1}{2}\sigma_X^2 \alpha_{2,k}^N(s)^2 + \beta^A \left(e^{k_x^c} - 1 \right),$$

with initial conditions, $\alpha_{0,k}^N(0) = \alpha_{2,k}^N(0) = 0$.

A.3.3 The case of a large economy: $N \to \infty$

In the case of a large diversified economy considered in Section 4.4, i.e. N large and $s_j = 1/N$, price expressions simplify since $u_j = 0 \quad \forall j$. The price-dividend ratio of any stock j is given by:

$$g_j(\phi, X, \mathbf{u}, t) = N^\gamma \sum_{k=0}^\gamma a_k(\phi) \int \mathcal{F}_\gamma^N(\mathbf{z}) b_{jk}(X, t, \mathbf{z}) d\mathbf{z}. \tag{79}$$

B Variance risk premium due to large jumps

Bollerslev and Todorov (2011) develop a nonparametric method to isolate the fraction of the observed variance risk premium due to large jumps. Since the model-implied variance risk premium only includes compensation for jump risk, it is useful, as a robustness check, to run the predictive regressions in Section 5 using Bollerslev and Todorov (2011)'s time series of market variance risk premium only due to large jumps (VRP^j), which is available from February 1996 through July 2007. Figure 21 compares this measure of the variance premium only due to jumps with the variance risk premium measure used in the main text. The correlation between the two series is about 73%. I first consider the predictive regression

$$r_{t+6}^e = \alpha + \beta VRP_t^j + \varepsilon_{t+6}, \tag{80}$$

where r_{t+6}^e is the excess return of the $S\&P500$ index at a 6 months horizon. Figure 22 shows regression coefficient estimates with 95% confidence bounds (upper panel) and adjusted R^2 in percentage (upper panel) estimated on a rolling window of 50 months. Consistent with the results in the main text, predictive power is stronger in phases of large disagreement, such as in the early 2000 and at the onset of the recent financial crisis. Then, analogously to Figure 15, Figure 23 shows the distributions of regression coefficients (upper panel) and R^2 (lower panel), for small, average, and large values of the difference in beliefs, obtained applying a bock bootstrap procedure. Both the regression coefficient and the adjusted R^2 increase (in absolute value) with the level of DB. Results are a bit less strong than what I find using

the aggregate variance risk premium in the man text, but they have to be taken with caution since the number of observations in every bin is small.[44]

Then I estimate regressions of the form:

$$r^e_{i,t+6} = \alpha_i + \beta_i VRP^j_t + \varepsilon_{t+6}, \tag{81}$$

for $i = S, M, B$, where $r^e_{i,t+h}$ is the monthly excess returns on small-, mid-, and big-cap portfolios, respectively, at the 6-month horizon. Consistent with the results in the main text, predictive power is stronger for small stocks, with an adjusted R^2 of 2.1% against an adjusted R^2 of -0.4% for large stocks.

C   Tables and Figures

Table 1: Model parameters

Preferences	$\delta = 0.04$	$\gamma = 4$	
Dividends	$\mu = 2.5\%$	$\sigma = 5\%$	$k = -0.41$
Intensities	$\beta^A = 1\%$	$\beta^B = 0.01\%$	$\lambda = 1\%$
	$\varphi = 0.142$	$\sigma_X = 0.05$	

Table 2: Simulated market variance risk premium

c^A	0.1	0.3	0.5	0.7	0.9
mean VRP	-4.70	-5.48	-6.76	-8.85	-12.74
	(1.13)	(1.33)	(1.50)	(1.52)	(1.33)
std VRP	0.55	0.66	0.82	0.99	0.95
	(0.27)	(0.47)	(0.65)	(0.71)	(0.42)

---

[44]Since VRP^j is available only from February 1996 through July 2007, there are less than 50 monthly observations for each DB quantile.

Table 3: Market return predictability by variance risk premium (VRP), from simulated monthly data, at horizons $h = 1, 6$ and 12 months, for different values of the initial consumption share of the pessimistic agent, $c^A = 0.1$, 0.5, and 0.9. The table shows the average of the regression coefficient and adjusted R^2 over all simulations, with standard errors in parenthesis. Returns are in annualized percentage while VRP is in monthly squared percentage.

	$c^A = 0.1$			$c^A = 0.5$			$c^A = 0.9$		
Horizon (months)	1	6	12	1	6	12	1	6	12
VRP Coeff	-0.51	-0.20	-0.19	-0.38	-0.24	-0.24	-0.35	-0.26	-0.25
	(0.19)	(0.18)	(0.17)	(0.12)	(0.15)	(0.14)	(0.08)	(0.06)	(0.07)
Adj R^2 (%)	2.15	4.04	7.56	3.15	10.12	18.34	4.26	14.32	23.97
	(0.98)	(3.29)	(6.29)	(1.54)	(6.06)	(10.94)	(1.80)	(7.69)	(12.97)

Table 4: Predictability of excess returns of small (Panels A and C) and big (Panels B and D) stock by variance risk premium (VRP, Panels A and B) and by covariance risk premium (CRP, Panels C and D), from simulated monthly data, at horizons $h = 1$, 6, and 12 months, for different values of the initial consumption share of the pessimistic agent, $c^A = 0.1$, 0.5, and 0.9. The small (big) stock has an initial dividend share of $s = 0.1$ ($s = 0.9$). The table shows the average of the regression coefficient and adjusted R^2 over all simulations, with standard errors in parenthesis. Returns are in annualized percentage while VRP and CRP are in monthly squared percentage.

	Panel A: Regression of small stock returns on VRP								
	$c^A = 0.1$			$c^A = 0.5$			$c^A = 0.9$		
Horizon (months)	1	6	12	1	6	12	1	6	12
VRP Coeff	0.41	0.04	-0.02	0.25	0.06	0.02	0.02	-0.07	-0.08
	(0.38)	(0.26)	(0.25)	(0.34)	(0.24)	(0.22)	(0.21)	(0.23)	(0.23)
Adj R^2 (%)	1.57	4.02	7.13	1.30	4.13	6.67	0.72	5.86	9.61
	(1.58)	(5.75)	(9.43)	(2.04)	(6.62)	(10.01)	(1.32)	(7.63)	(12.84)

	Panel B: Regression of big stock returns on VRP								
	$c^A = 0.1$			$c^A = 0.5$			$c^A = 0.9$		
Horizon (months)	1	6	12	1	6	12	1	6	12
VRP Coeff	-0.54	-0.15	-0.14	-0.42	-0.24	-0.23	-0.37	-0.26	-0.25
	(0.24)	(0.20)	(0.19)	(0.14)	(0.15)	(0.14)	(0.09)	(0.07)	(0.07)
Adj R^2 (%)	1.91	2.43	4.53	3.00	8.45	15.27	3.96	12.46	21.27
	(1.06)	(3.62)	(5.02)	(1.48)	(5.77)	(10.50)	(1.74)	(7.24)	(12.48)

	Panel C: Regression of small stock returns on CRP								
	$c^A = 0.1$			$c^A = 0.5$			$c^A = 0.9$		
Horizon (months)	1	6	12	1	6	12	1	6	12
CRP Coeff	-0.38	-0.37	-0.36	-0.31	-0.32	-0.32	-0.32	-0.30	-0.28
	(0.11)	(0.10)	(0.11)	(0.13)	(0.13)	(0.13)	(0.06)	(0.07)	(0.08)
Adj R^2 (%)	1.95	11.37	19.09	1.58	9.78	16.28	4.14	18.84	28.22
	(0.96)	(5.05)	(8.16)	(1.50)	(7.27)	(11.43)	(2.79)	(11.05)	(15.34)

	Panel D: Regression of big stock returns on CRP								
	$c^A = 0.1$			$c^A = 0.5$			$c^A = 0.9$		
Horizon (months)	1	6	12	1	6	12	1	6	12
CRP Coeff	0.14	0.14	0.13	-0.07	-0.04	-0.04	-0.18	-0.16	-0.15
	(0.15)	(0.15)	(0.16)	(0.23)	(0.23)	(0.25)	(0.19)	(0.15)	(0.14)
Adj R^2 (%)	0.36	3.43	7.41	0.51	4.03	8.57	1.19	6.04	10.78
	(0.82)	(4.15)	(8.15)	(1.03)	(5.67)	(9.99)	(1.39)	(6.05)	(10.90)

Table 5: Survival. This table displays the share of consumption of the optimistic agent, c^B, at horizon $T = 50$, 100, and 500 years, obtained from 1,000 simulations starting from $c^B = 0.1$, 0.5, and 0.9.

c^B	$T = 50$	$T = 100$	$T = 500$
0.1	0.08	0.07	0.02
	(0.03)	(0.04)	(0.04)
0.5	0.44	0.38	0.14
	(0.12)	(0.15)	(0.16)
0.9	0.86	0.81	0.44
	(0.08)	(0.13)	(0.28)

Table 6: Summary statistics: variance risk premium.

	Full Sample			Pre-crisis		
	IV	ERV	VRP	IV	ERV	VRP
Mean	40.33	21.74	-18.60	33.14	16.65	-16.49
Median	31.66	15.72	-14.42	24.62	12.05	-13.23
Std.dev	36.35	25.69	14.21	23.99	13.04	11.99
Max	298.90	282.68	-4.01	163.39	73.13	-4.01
Min	9.05	3.99	-91.16	9.05	3.99	-91.16
Skew	3.24	5.44	-2.27	2.01	1.97	-2.35
Kurt	18.18	47.31	9.63	8.98	7.61	11.75
AC(1)	0.81	0.76	0.63	0.79	0.81	0.68

Table 7: Market return predictability by variance risk premium

Panel A: Full sample (1990.1-2011.12)					
Horizon (months)	1	3	6	9	12
VRP Coeff	-0.553	-0.555	-0.476	-0.342	-0.272
t-stat	-2.048	-2.777	-3.705	-2.991	-2.702
Adj R^2 (%)	1.88	5.99	7.75	5.74	4.63
Panel B: Pre-crisis sample (1990.1-2007.7)					
Horizon (months)	1	3	6	9	12
VRP Coeff	-0.509	-0.557	-0.307	-0.176	-0.119
t-stat	-2.435	-3.160	-2.005	-1.191	-0.922
Adj R^2 (%)	1.19	5.83	3.59	1.48	0.56
Panel C: Robust, Full sample (1990.1-2011.12)					
Horizon (months)	1	3	6	9	12
VRP Coeff	-0.824	-0.649	-0.473	-0.353	-0.292
t-stat	-4.021	-5.896	-5.338	-5.060	-4.860
Adj R^2 (%)	1.71	6.16	8.11	6.10	4.98
Panel D: Robust, Pre-crisis sample (1990.1-2007.7)					
Horizon (months)	1	3	6	9	12
VRP Coeff	-0.694	-0.554	-0.332	-0.244	-0.193
t-stat	-2.774	-4.238	-3.363	-3.102	-2.804
Adj R^2 (%)	1.44	6.28	4.02	1.60	0.64

Table 8: Single stock returns predictability by variance risk premium for the full sample (1990.1–2011.12). Stocks are sorted into quintile portfolios based on their estimated VRP loading, β_i. Panels A-E report average statistics in each quintile.

Panel A: First Quintile					
Horizon (months)	1	3	6	9	12
VRP Coeff	-1.127	-1.431	-1.411	-1.097	-0.870
t-stat	-1.543	-2.452	-2.964	-2.798	-2.556
Adj R^2 (%)	1.01	5.76	10.37	10.38	9.54
Panel B: Second Quintile					
Horizon (months)	1	3	6	9	12
VRP Coeff	-0.696	-0.781	-0.686	-0.486	-0.360
t-stat	-1.372	-2.024	-2.341	-2.069	-1.818
Adj R^2 (%)	0.61	2.95	4.66	3.99	3.25
Panel C: Third Quintile					
Horizon (months)	1	3	6	9	12
VRP Coeff	-0.567	-0.525	-0.433	-0.295	-0.241
t-stat	-1.222	-1.726	-1.862	-1.559	-1.493
Adj R^2 (%)	0.41	1.64	2.42	2.12	2.32
Panel D: Fourth Quintile					
Horizon (months)	1	3	6	9	12
VRP Coeff	-0.250	-0.295	-0.194	-0.147	-0.105
t-stat	-0.582	-1.021	-0.929	-0.894	-0.727
Adj R^2 (%)	0.04	0.58	0.48	0.62	0.63
Panel E: Fifth Quintile					
Horizon (months)	1	3	6	9	12
VRP Coeff	0.114	0.003	0.071	0.063	0.077
t-stat	0.314	0.011	0.347	0.351	0.508
Adj R^2 (%)	-0.14	-0.16	-0.03	0.15	0.61

Table 9: Return predictability by variance risk premium for CRSP cap-based portfolios. Panel A include regression estimates at the one-month horizon, Panel B is for the six-month horizon and Panel C for the 12-month horizon.

Panel A: one-month horizon			
Portfolio	Big Cap	Mid Cap	Small Cap
VRP Coeff	-0.527	-0.597	-0.630
t-stat	-1.975	-2.075	-1.820
Adj R^2 (%)	1.69	1.45	1.12
Panel A: six-month horizon			
Portfolio	Big Cap	Mid Cap	Small Cap
VRP Coeff	-0.435	-0.573	-0.736
t-stat	-3.335	-3.844	-3.720
Adj R^2 (%)	6.21	8.19	10.17
Panel A: 12-month horizon			
Portfolio	Big Cap	Mid Cap	Small Cap
VRP Coeff	-0.229	-0.354	-0.450
t-stat	-2.310	-3.099	-3.237
Adj R^2 (%)	2.86	7.28	9.67

Table 10: Return predictability by variance risk premium for small-, mid-, and big-cap portfolios from CRSP, at the 6-month horizon, for different levels of the difference in beliefs. For comparison, the last two panel report results of the same predictive regression for the $S\&P500$ index return and for small-minus big-cap portfolio return, respectively.

		Small DB	Average DB	Large DB
small-cap	VRP Coeff	0.06	-0.32	-1.37
	Adj R^2 (%)	-1.36	2.87	21.26
		Small DB	Average DB	Large DB
mid-cap	VRP Coeff	0.19	-0.29	-1.11
	Adj R^2 (%)	-0.41	3.82	17.79
		Small DB	Average DB	Large DB
big-cap	VRP Coeff	-0.18	-0.29	-0.82
	Adj R^2 (%)	-0.04	2.86	14.41
		Small DB	Average DB	Large DB
$S\&P500$	VRP Coeff	-0.24	-0.31	-0.97
	Adj R^2 (%)	1.09	4.02	20.91
		Small DB	Average DB	Large DB
small-big	VRP Coeff	0.17	0.11	-0.60
	Adj R^2 (%)	-0.55	-0.35	21.42

Table 11: Mean and standard deviation of returns (in annualized percentage) for small- and big-cap portfolios from CRSP, at the 6-month horizon, for different levels of the difference in beliefs.

		Small DB	Average DB	Large DB
small-cap	Mean (%)	5.40	18.17	11.37
	Standard Deviation (%)	21.54	22.90	45.48
		Small DB	Average DB	Large DB
big-cap	Mean (%)	7.94	14.53	2.04
	Standard Deviation (%)	15.00	19.62	32.61

Figure 1: Instantaneous equity premium of stock 1, under agent A's beliefs, as a function of the dividend share of asset 1, s_1, when the consumption share of the pessimistic agent is $c^A = 0.1$ (left panel) or $c^A = 0.9$ (right panel).
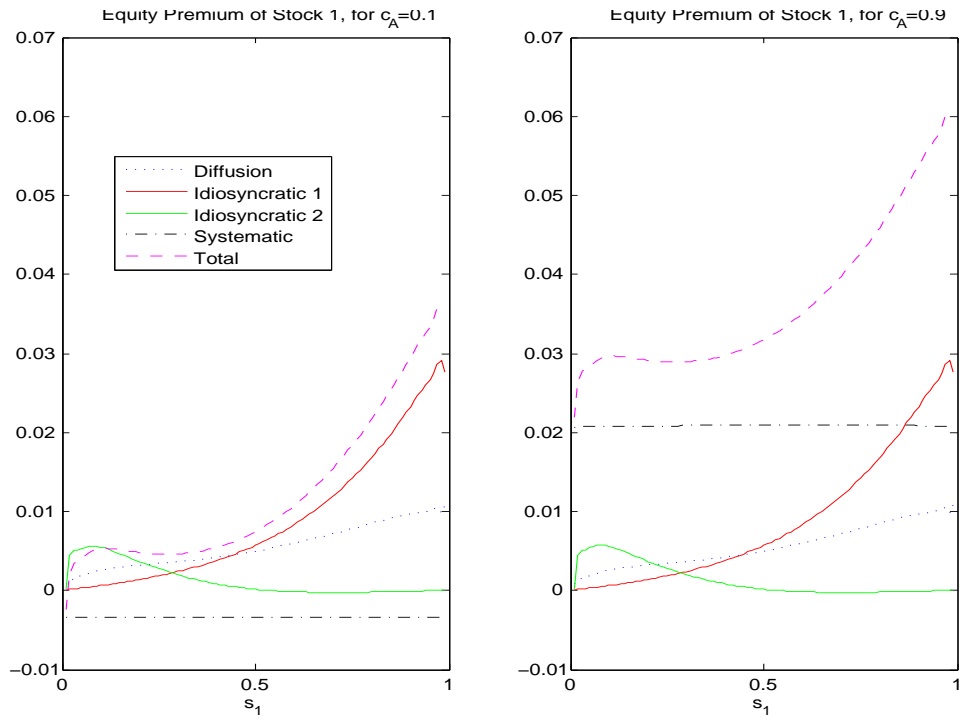
Figure 2: The left panel plots the systemic jump premium, JP_c, as a function of agent A's consumption. The right panel plots the idiosyncratic jump premium, JP_j, as a function of the dividend share of asset j.
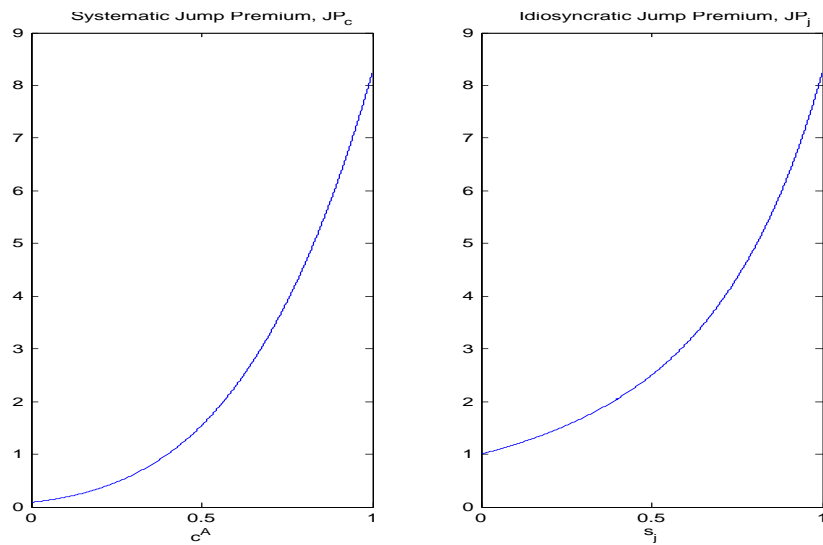
Figure 3: The left panel plots the jump size in the return of a stock at a systemic disaster, $k_{S_j}^c$, as a function of the consumption share of the pessimistic agent, c^A, for a small stock ($s = 0.1$) and a large stock ($s = 0.9$), as well as in the case of no disagreement. The right panel displays the jump size in stock 1's return at an idiosyncratic jump in its dividend growth process (blue line), $k_{S_1,1}$, and in the dividend growth process of the second asset (red line), $k_{S_1,2}$, respectively.
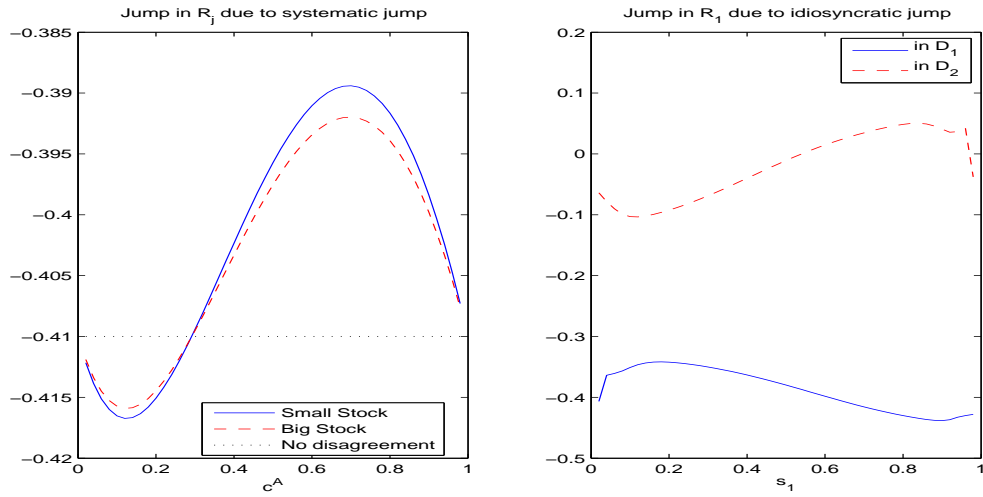
Figure 4: Instantaneous variance risk premium of stock 1, under agent A's beliefs, in monthly squared percentage, as a function of the dividend share of asset 1, s_1, for different values of the consumption share of the pessimistic agent is c^A. The second, third and fourth panels show the decomposition of the individual variance risk premium in its idiosyncratic and systemic jump components when the consumption share of the pessimistic agent is $c^A = 0.1$, $c^A = 0.5$ and $c^A = 0.9$, respectively.
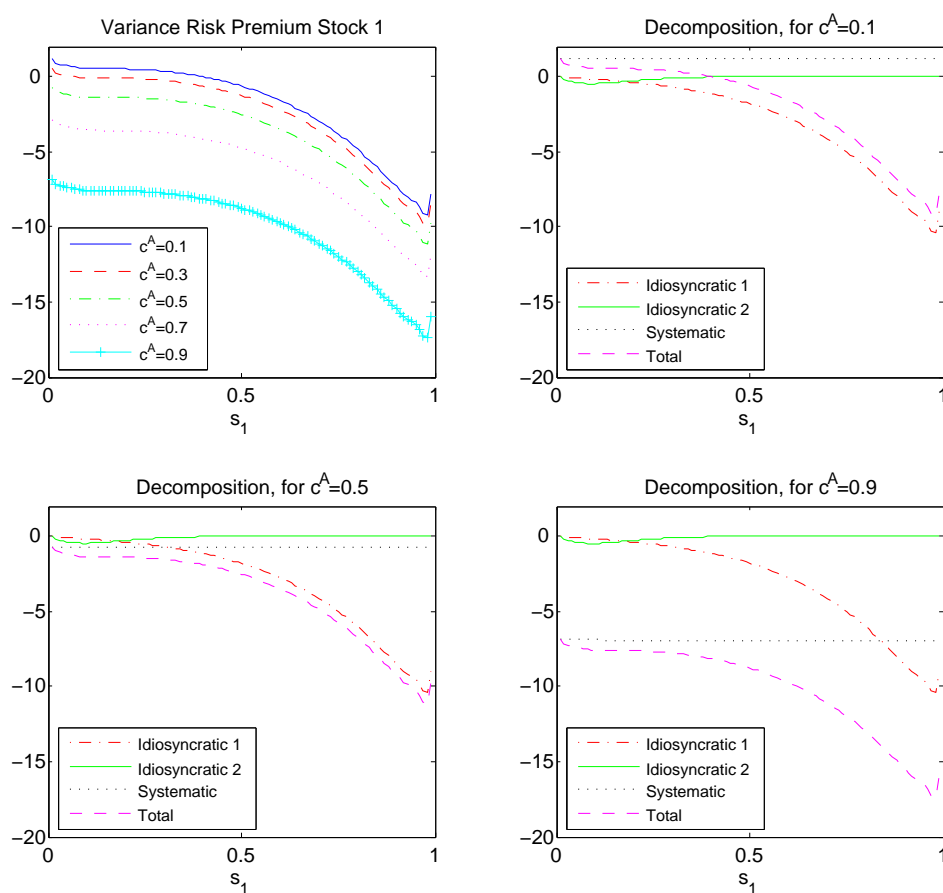
Figure 5: Instantaneous equity (upper panels, in percentage) and variance (lower panels, in monthly squared percentage) risk premium of the market, under agent A's beliefs, as a function of the dividend share of asset 1, s_1, and their decomposition in terms of individual equity and variance premia, for different values of the consumption share of the pessimistic agent is c^A. The first and third panels use $c^A = 0.1$, while the second and the fourth are for $c^A = 0.9$.
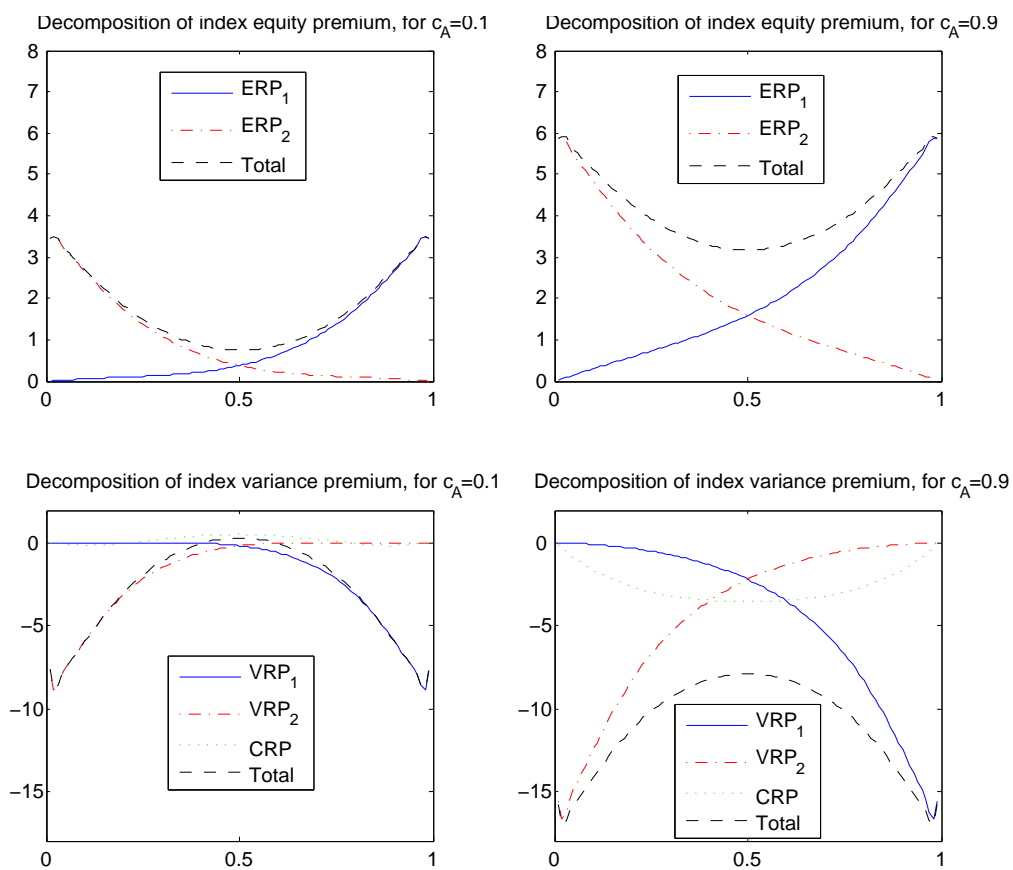
Figure 6: Instantaneous conditional stock return correlation in an economy with $N = 2$ assets.
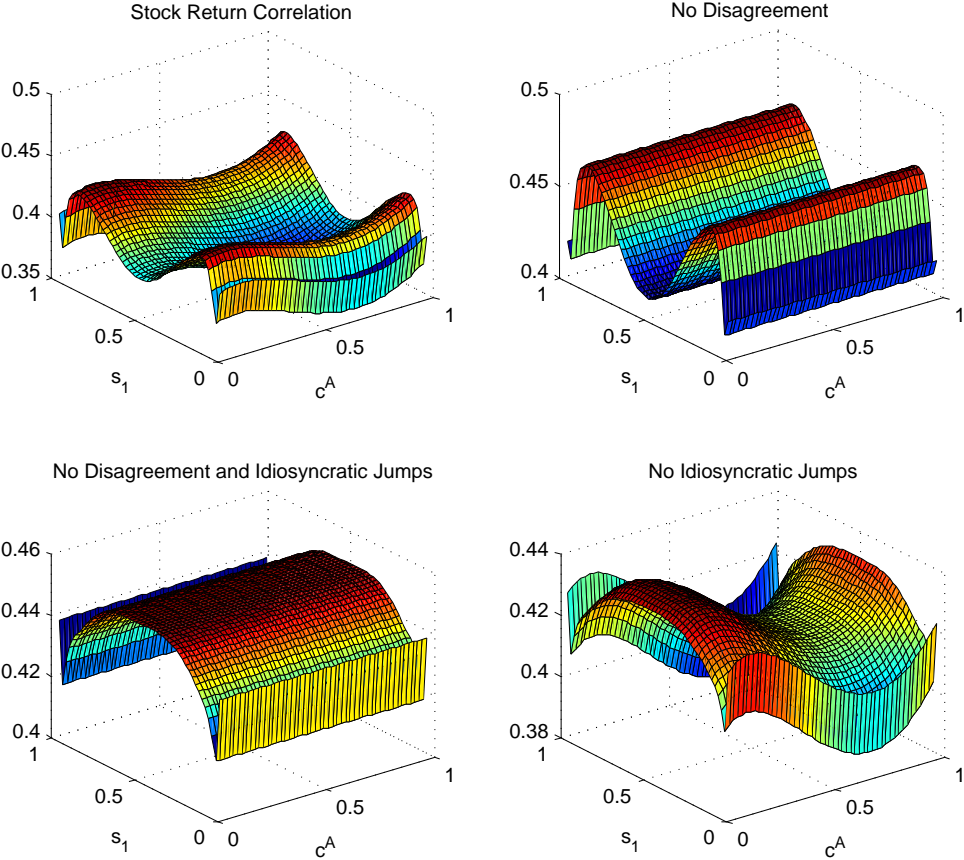
Figure 7: Instantaneous conditional stock return correlation under the risk-neutral measure, in an economy with $N = 2$ assets.

Risk Neutral Correlation

No Disagreement

No Disagreement and Idiosyncratic Jumps

No Idiosyncratic Jumps
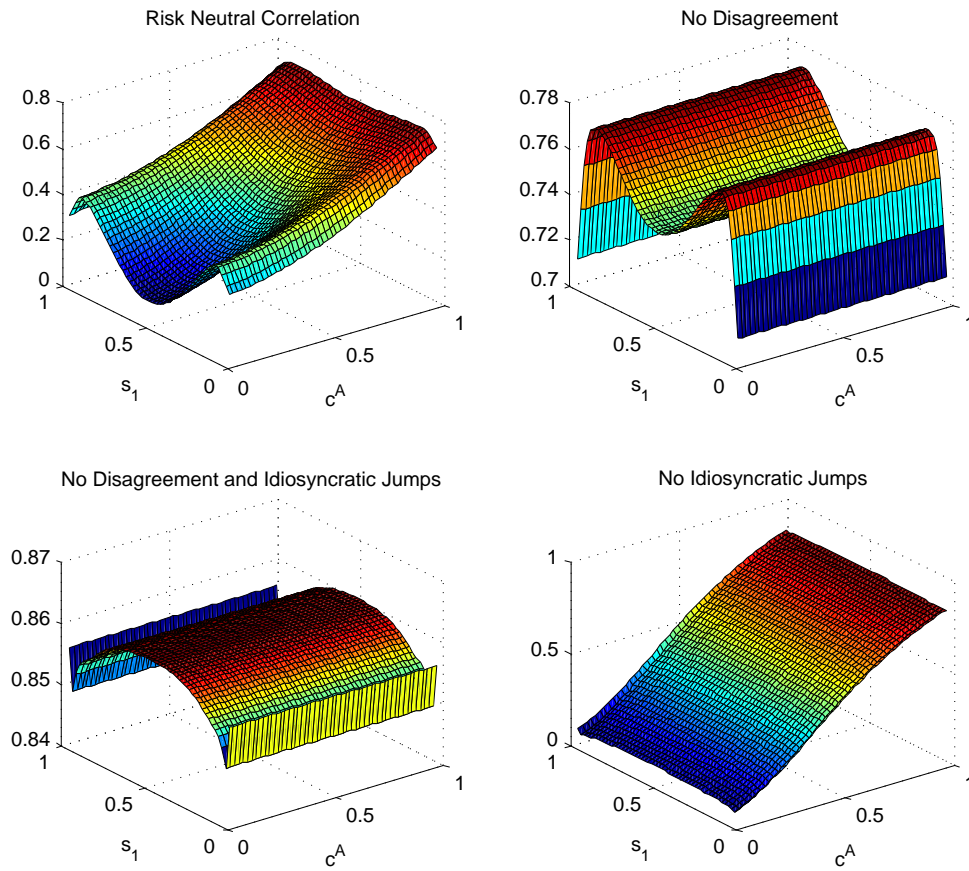
Figure 8: Instantaneous correlation risk premium, in an economy with $N = 2$ assets.

Correlation Risk Premium
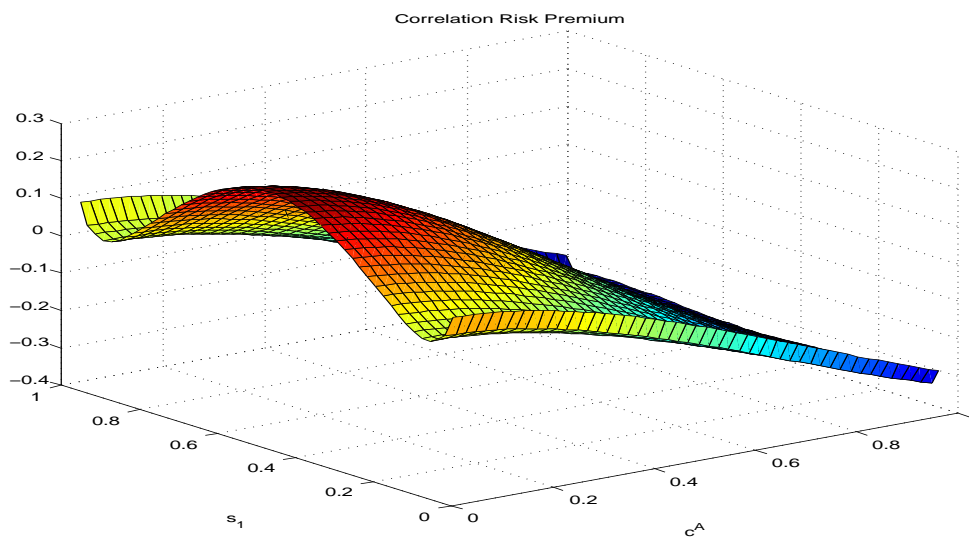
Figure 9: Standard OLS regression of simulated excess market returns at the six-month horizon on the simulated lagged instantaneous variance risk premium, for different levels of the initial share of consumption of the pessimistic agent, c^A. Upper panel display the distribution of simulated regression coefficients and lower panel of percentage R^2.
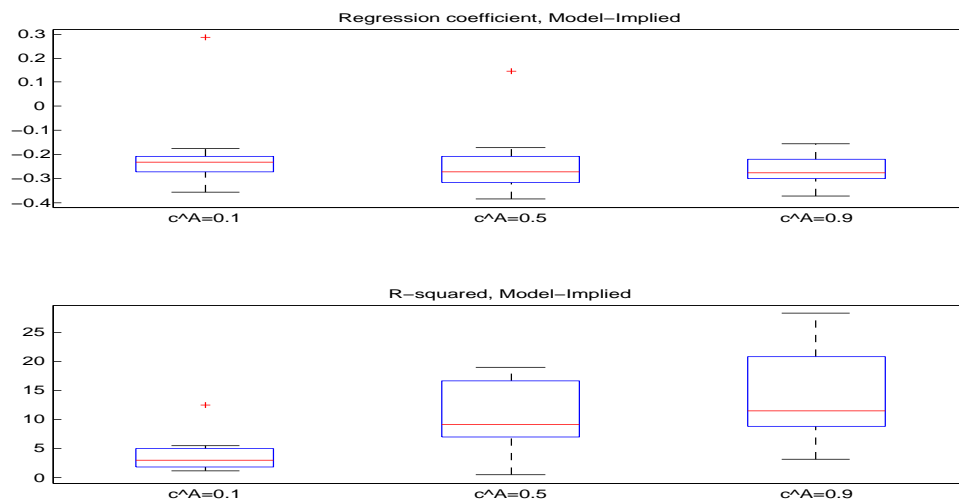
Figure 10: Standard OLS regression of simulated excess returns at the six-month horizon of a small stock (starting from $s = 0.1$, blue box plots) and a big stock (starting from $s = 0.9$, red box plots) on the simulated lagged instantaneous covariance risk premium, for different levels of the initial share of consumption of the pessimistic agent, c^A. Upper panel display the distribution of simulated regression coefficients and lower panel of percentage R^2.
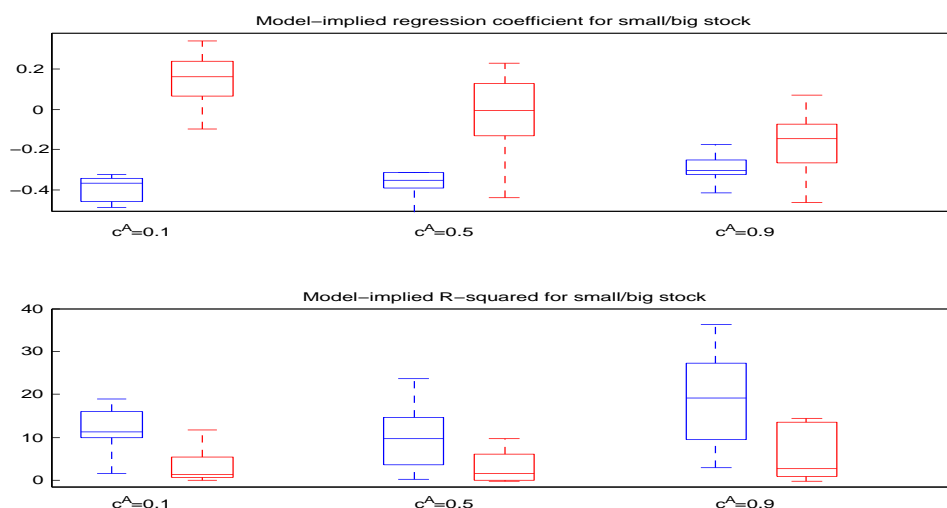
Figure 11: Instantaneous equity premium (annualized and in percentage, first panel),systemic jump premium (second panel) and index variance risk premium (in monthly terms and squared percentage, third panel), under agent A's beliefs, in the case of a large symmetric economy, as a function of the consumption share of the pessimistic agent, c^A.
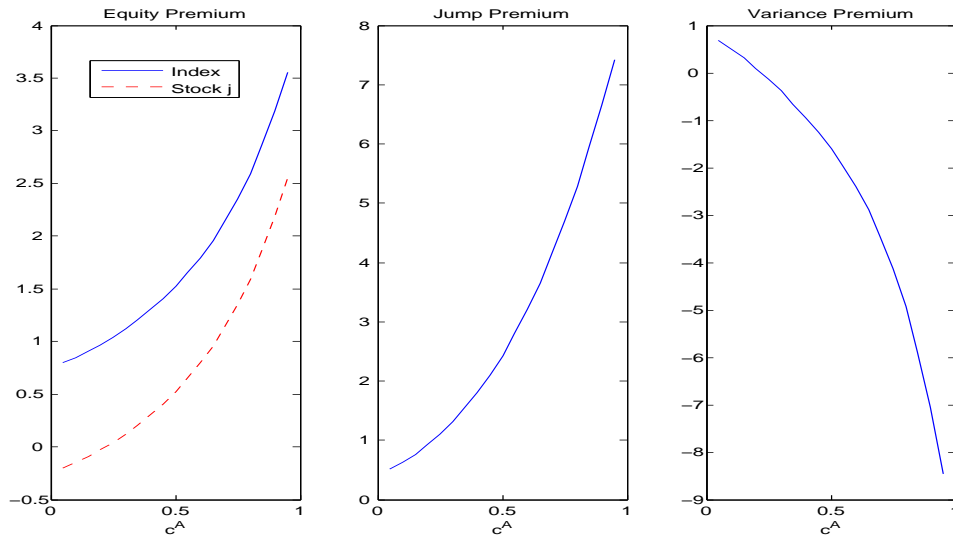
Figure 12: Size of jumps in stock returns due to a systemic disaster, k_s^c, under agent A's beliefs, in the case of a large symmetric economy, as a function of the consumption share of the pessimistic agent, c^A.
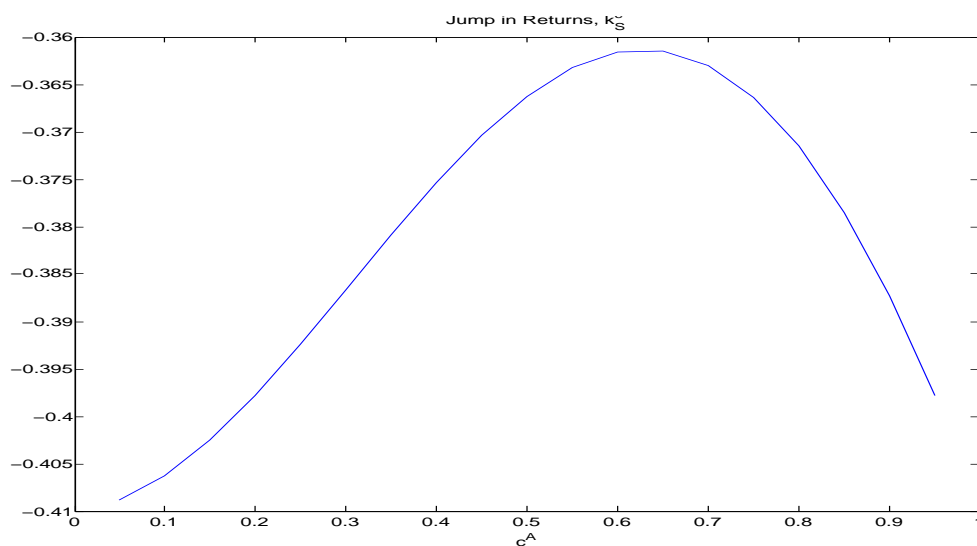
Figure 13: Time series of variance risk premium, in monthly squared percentage, where the physical expectation of the realized variance is computed from a projection of realized variance on the value of the lagged squared VIX and on lagged realized variance. Light gray shaded areas denote phases in which difference in beliefs, measured based on the dispersion of one-year-ahead forecasts on real GDP growth from the BlueChip Economic Indicator, is above average. Dark gray shaded areas denote NBER recessions.
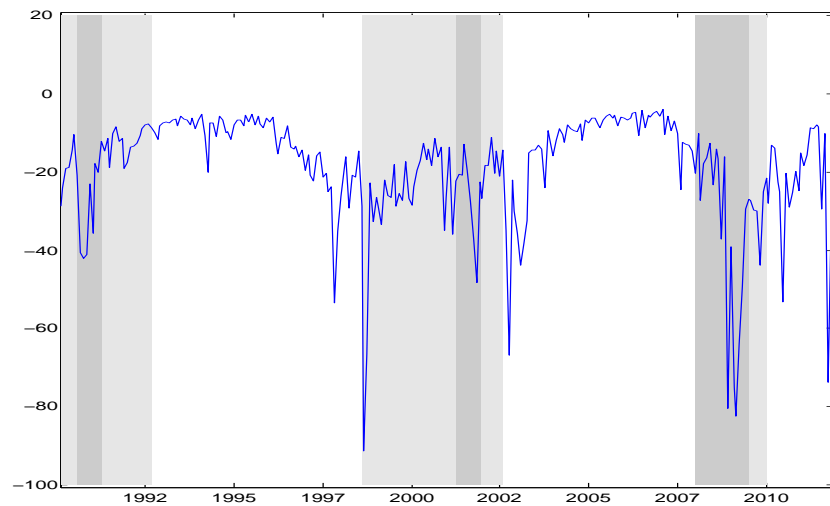
Figure 14: Standard OLS regression of excess market returns at the six-month horizon on the lagged variance risk premium, for different levels of the VRP. The first box plot corresponds to small absolute values of the premium ($VRP < q_{70\%}$), the last to large values ($VRP > q_{30\%}$) and the middle box plot to average values of the VRP. Upper panel display the distribution of regression coefficients and lower panel of percentage R^2, both obtained applying a block bootstrap procedure.
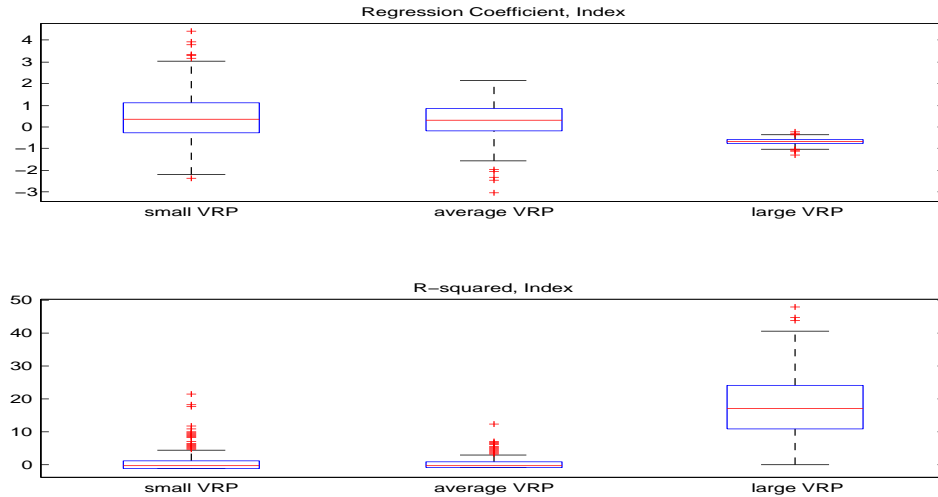
Figure 15: Standard OLS regression of excess market returns at the six-month horizon on the lagged variance risk premium, for different levels of the difference in beliefs (DB). The first box plot corresponds to small values of disagreement (DB< $q_{30\%}$), the last to large values (DB> $q_{70\%}$) and the middle box plot to average values of DB. Upper panel display the distribution of regression coefficients and lower panel of percentage R^2, both obtained applying a block bootstrap procedure.
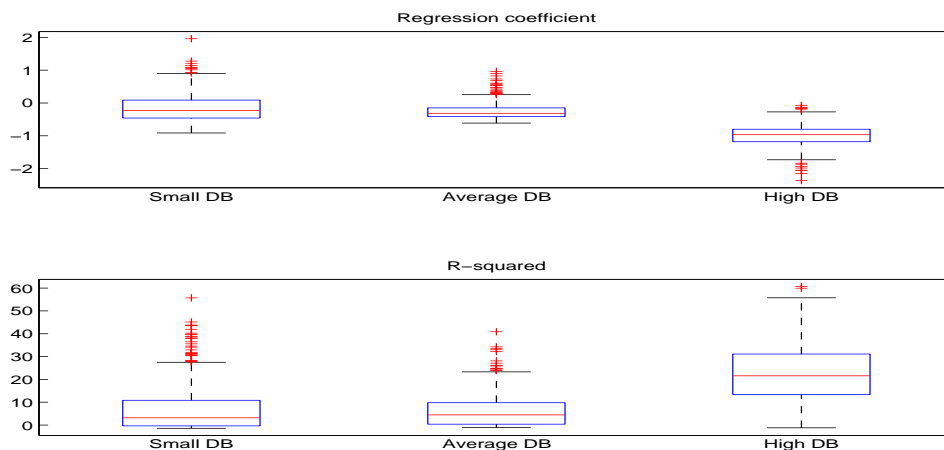
Figure 16: Kernel regression of standardized excess market returns at the six-month horizon, in annualized percentage, on the lagged variance risk premium, in monthly squared percentage. Single dots represent the data, while the solid line is an estimated kernel regression using Nadaraya–Watson estimator with a Gaussian kernel.
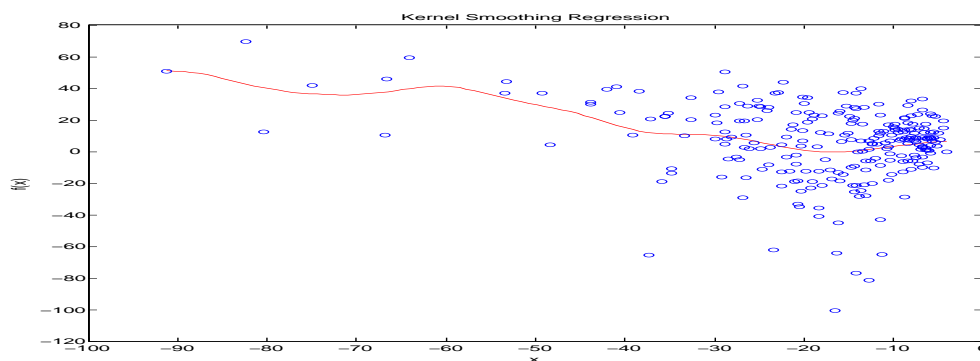

Figure 17: Predictive regressions of market 6-month excess returns on lagged variance risk premium on a rolling window of 50 months. Upper panel shows regression coefficient estimates with 95% confidence bounds, while lower panel reports adjusted R^2 in percentage. The dashed red line in the upper panel denotes the regression coefficient estimated on the full sample.
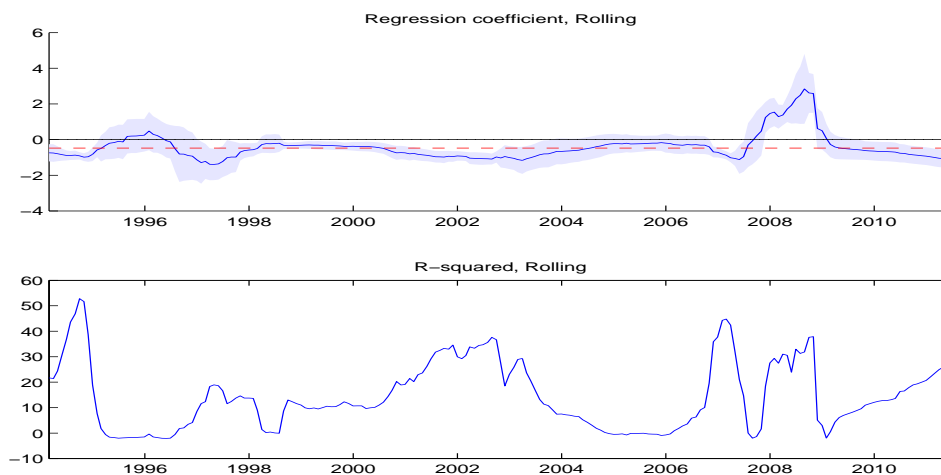
Figure 18: Time series of variance risk premia with estimated regimes. Shaded areas correspond to state 2, which is characterized by stronger return predictability.
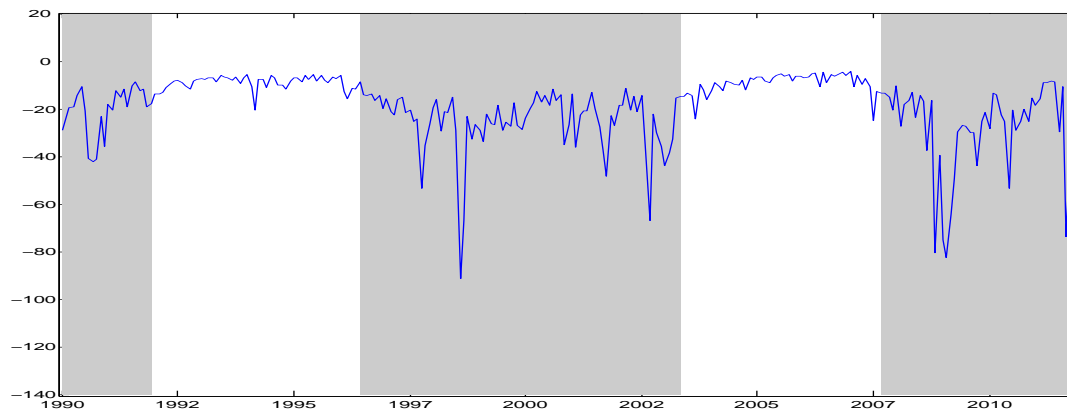

Figure 19: Standard OLS regression of excess returns of Fama and French portfolios at the six-month horizon on the lagged market variance risk premium. Left panel displays the regression coefficients and right panel the percentage R^2. Lines connect portfolios of different book-to-market categories within each size category, focusing on the bottom and upper quintiles, which correspond to small and big stocks, respectively.
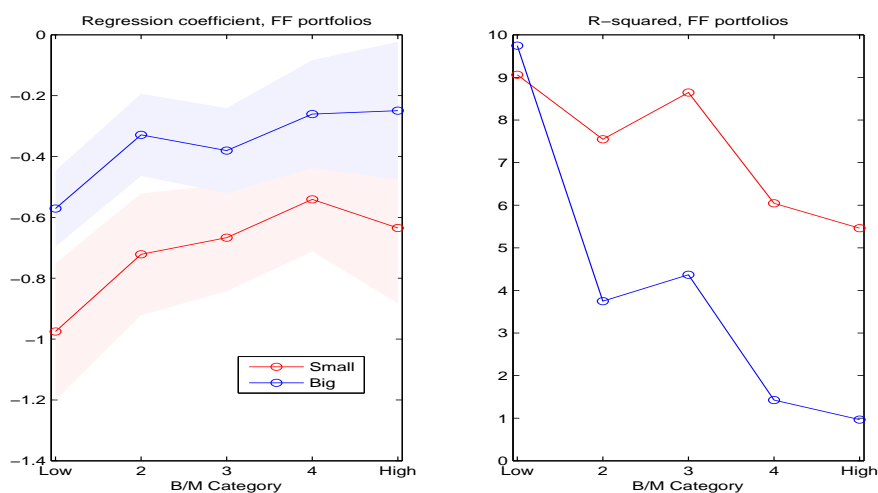
Figure 20: Standard OLS regression of excess returns at the six-month horizon on the lagged market variance risk premium for the CRSP cap-based portfolios, for different levels of the difference in beliefs proxy, DB. Left panels display the distribution of regression coefficients and left panels of adjusted R^2, both obtained applying a block bootstrap procedure.
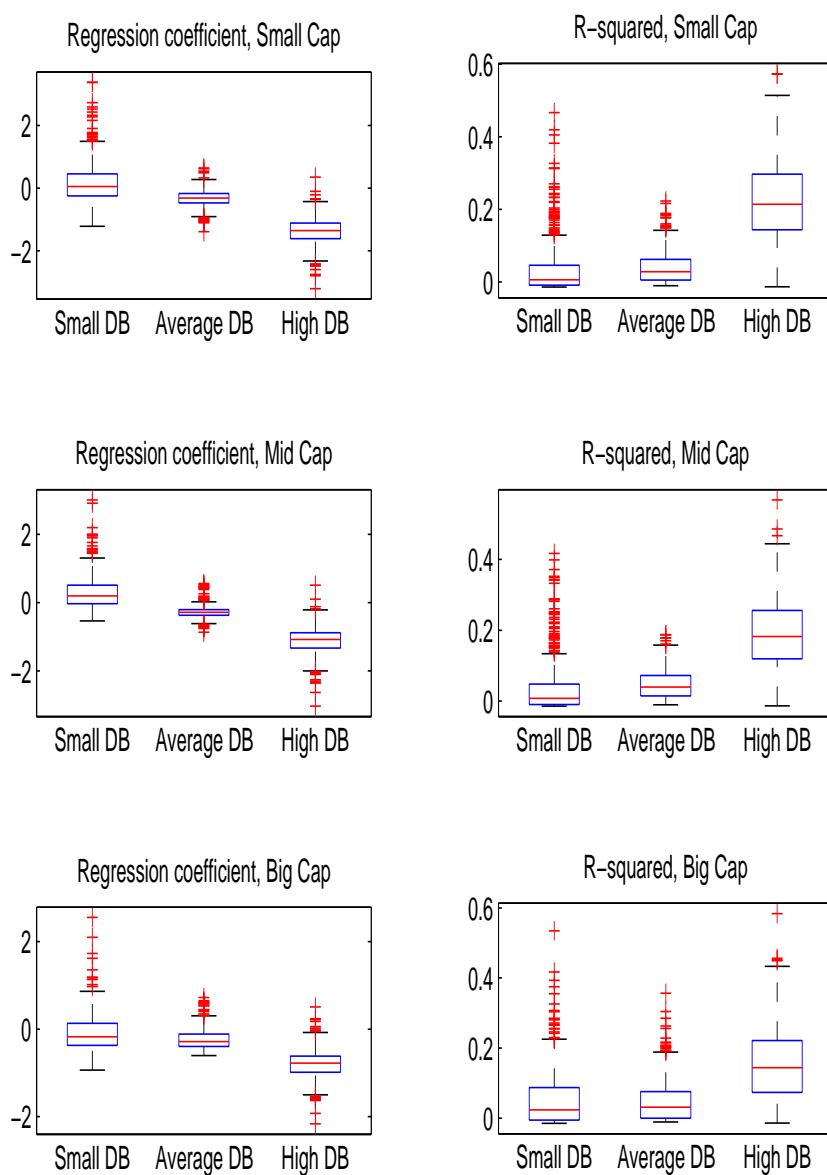
Figure 21: Time series of variance risk premium only due to jumps, from Bollerslev and Todorov (2011), versus the total variance risk premium measure described in Section 5.1, for the overlapping sample, that goes from February 1996 through July 2007. Both measures are in monthly squared percentage.
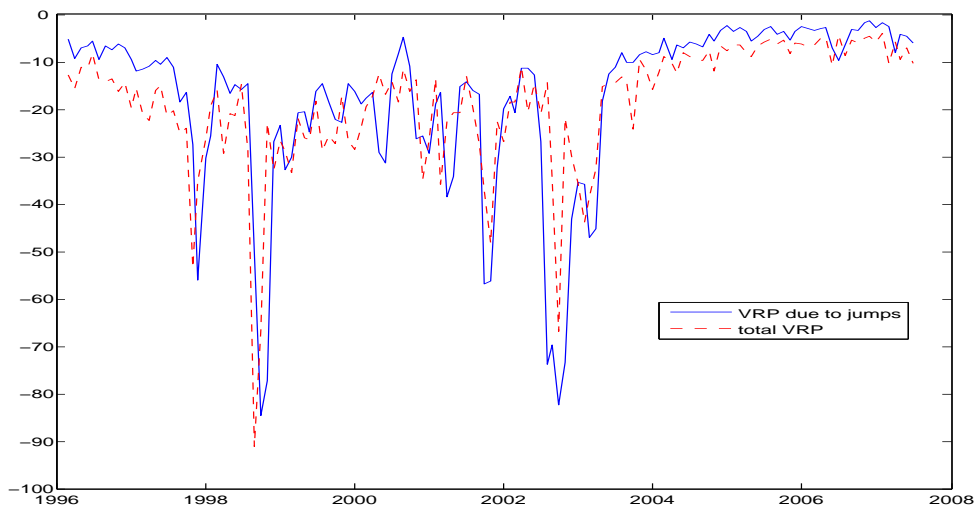

Figure 22: Predictive regressions of market 6-month excess returns on lagged variance risk premium only due to jumps, from Bollerslev and Todorov (2011), on a rolling window of 50 months. Upper panel shows regression coefficient estimates with 95% confidence bounds, while lower panel reports adjusted R^2 in percentage.
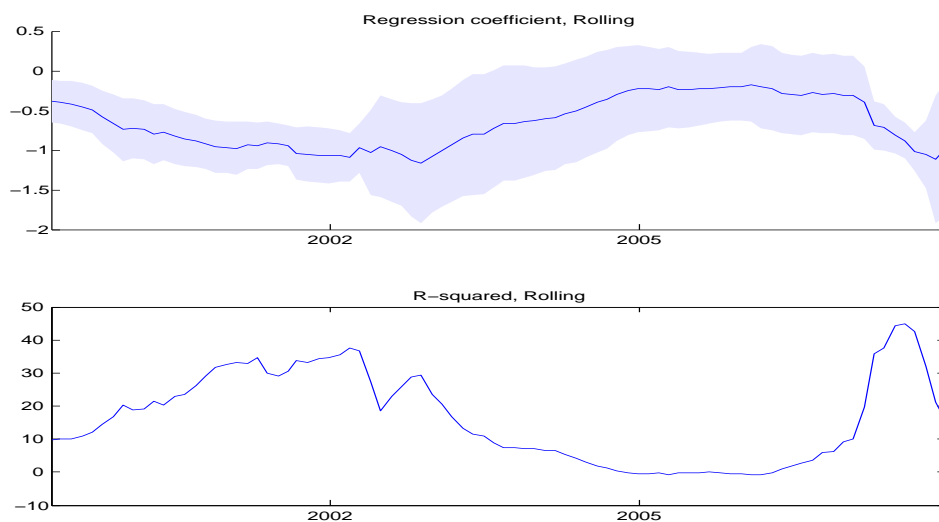
Figure 23: Standard OLS regression of excess market returns at the six-month horizon on the lagged variance risk premium only due to jumps, from Bollerslev and Todorov (2011), for different levels of the difference in beliefs (DB). The first box plot corresponds to small values of disagreement (DB$< q_{30\%}$), the last to large values (DB$> q_{70\%}$) and the middle box plot to average values of DB. Upper panel display the distribution of regression coefficients and lower panel of percentage R^2, both obtained applying a block bootstrap procedure.