Money Illusion:

A Rationale for the TIPS Puzzle

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Abstract

Why is the TIPS market so small? We show that a rational individual, dynamically investing into multiple asset classes over a 20-year horizon, benefits by 1.2% per annum from having access to inflation-indexed bonds. However, if the investor suffers from money illusion, the perceived certainty equivalent gains reduce to less than 0.3%. Furthermore, the benefits become totally negligible if the money-illusioned investor is less sophisticated and ignores time variations in risk premia. Money illusion causes significant portfolio shifts from inflation-indexed toward nominal bonds, with little effects on equity allocations.

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1 Introduction

Introduced in the U.S. in 1997, after several periods of high inflation uncertainty, Treasury Inflation-Protected Securities (TIPS) should have been acclaimed by market participants. They were supposed to be a close-to-perfect instrument to hedge against inflation and would have offered an observable measure of the term structure of inflation expectations. Early works in asset allocation strongly supported the welfare improvement entailed by the inclusion of such securities (Campbell and Viceira, 2001; Brennan and Xia, 2002; Wachter, 2003; Kothari and Shanken, 2004). However, a few years later, it was obvious that the TIPS market had not delivered in terms of market quality, as the level of market illiquidity was still noticeably high, especially relative to nominal Treasury Bonds. Since then, instead of improving, the illiquidity of the TIPS market has persisted, if not deteriorated (Fleming and Krishnan, 2012). Although the illiquidity does not seem to question the cheapness of these securities, when compared to other types of debt financing (Campbell and Viceira, 2009), Figure 1 reveals that the fraction of TIPS with respect to the total of marketable Treasuries was 8.9% at the end of 2016, and has never been higher than 11% in the last two decades. The situation in other markets is not very different. For example, at the end of 2016, only 12.3% of the French public debt and only 12.7% of the Italian public debt was represented by inflation-protected securities. Among the largest issuers, the only exception seems to be represented by the UK, where inflation-protected debt has been issued since 1981 and represented 27.3% of the total outstanding debt in 2016.

[Figure 1 about here.]

While substantial progress has been made in identifying the TIPS liquidity premium, few works have focused on understanding *why* investors shun this asset class.[1] In this paper, we suggest that money illusion is a potential explanation for the failure of the TIPS market. Money illusion can be

[1]As a matter of fact, none of the inflation-related instruments have encountered the expected success. In addition to TIPS and inflation swaps, CPI-related futures were also introduced in 2004 by the Chicago Mercantile Exchange (CME), but failed to attract investors. Fleming and Sporn (2013) documented a very low trading activity on the inflation swap market, consisting of just over two trades per day on average since 2010.

described in general terms as a bias in the assessment of the real value of economic transactions, caused by the fact that people tend to think in nominal amounts.[2] The phenomenon of money illusion has been studied for more than half a century in monetary economics (Marschak, 1950, 1974; Dusansky and Kalman, 1974), as well as behavioral and experimental economics (Shafir et al., 1997; Fehr and Tyran, 2001, 2007, 2014), and has been found to be useful in solving several asset pricing puzzles.

We look at the distortions on the demand side, and the welfare consequences, brought about by money illusion in a market where multiple asset classes (nominal and indexed bonds, stocks and cash) are available. A conservative investor is first considered, who we formally model as an infinitely risk-averse investor. It is well known (Campbell and Viceira, 2001; Brennan and Xia, 2002) that a conservative investor, not suffering from money illusion, would allocate all her wealth to an indexed bond maturing at her investment horizon, or, if not available, would try to replicate it. However, when money illusion is at play, we find that the investor chooses a combination of indexed and nominal bonds, for which the relative weights depend on the degree of money illusion. Under extreme money illusion, all the portfolio is invested in a nominal bond maturing at the investment horizon. Severe money illusion therefore drives conservative investors out of the indexed bond market. While non-illusioned investors attempt to hedge future variations of the real rate, conservative money-illusioned investors partly or completely ignore expected inflation, hedging only variations of the nominal rate. When investors instead have a moderate risk aversion, the optimal allocation takes also advantage of the risk/return trade-off offered by indexed and nominal bonds, beyond their capacity to hedge inflation. We consider both the cases where risk premia are constant and time-varying, discovering that, again, money illusion significantly shifts the portfolio from indexed to nominal bonds. The impact of money illusion on stock investments is negligible if compared to the effect on optimal bond positions, as the inflation-hedging properties of both nominal and inflation-indexed bonds are

[2] In this work, money illusion can be either considered as a manifestation of irrationality, which might be the case for a significant fraction of individuals in the economy, or as a rational choice of the agent, which is the case, for example, for fund managers whose compensation is based on nominal returns.

quantitatively more relevant than for stocks.

A key contribution of our work is that we quantify the economic impact of money illusion under different perspectives. Firstly, we assess the opportunity cost entailed by money illusion, evaluating the expected utility loss of a non-illusioned investor who is forced to follow the portfolio strategy of a partially or totally money-illusioned investor. We find the loss to be substantial, with a certainty equivalent reduction of about 1% per annum for investment horizons longer than 10 years. This result is in line with the recent findings by Stephens and Tyran (2016), which were based on Danish financial and socio-demographic data. Secondly, we provide an explanation for the low market interest in inflation-protected bonds, by quantifying the opportunity cost, as perceived by both a non-illusioned investor and by a money-illusioned investor, of removing inflation-indexed bonds from the investable universe. Consistent with Mkaouar et al. (2017), we find that a non-illusioned investor, who ignores time variations in risk premia, suffers from a significant utility cost for not having access to inflation-indexed bonds. The certainty equivalent loss is about 0.5% per annum for a 10-year investment horizon and 1.25% per annum for a 30-year horizon. However, a money-illusioned investor perceives a loss inferior to 0.1% per annum when inflation-indexed bonds are not accessible. The losses are only slightly higher when investors are more sophisticated and account for time variations in risk premia, but the conclusion is the same: the utility loss suffered by a money-illusioned investor deprived of the access to inflation-protected instruments is very small, as her perceived expected utility achievable by substituting real bonds with nominal bonds is almost unchanged. It thus seems that money-illusioned investors have little incentives to enter the inflation-indexed bond market.

The most general framework we propose is based on a dynamic affine term structure model with time-varying risk premia, as in Dai and Singleton (2000), Duffee (2002), Joslin et al. (2011), and many others. The model is extended to allow for the pricing of inflation-indexed bonds, partially inspired by the more recent works by Christensen et al. (2010) and Andreasen et al. (2017). The term structure model is then applied to an asset allocation problem with multiple asset classes, as for example in Sangvinatsos and Wachter (2005). We contribute to the literature of dynamic asset allocation relying on dynamic term structure models, providing a different perspective to the empirical issues of the sensitivity to estimation errors and the in-sample overfitting of essentially affine models, as recently pointed out by Duffee (2011), Feldhütter et al. (2012) and Sarno et al. (2016). In this respect, we implement a simple but effective methodology, aimed at robustifying the model against the overfitting of time-varying risk premia. This problem typically leads to unrealistic optimal portfolio positions and largely overstates utility losses entailed by suboptimal strategies, which we avoid, in the estimation phase, by imposing reasonable bounds to the volatility of the risk premia in the economy.

As modern economies experienced low inflation rates during the Great Moderation years (80s, 90s) and also since the Great Recession (starting with the financial crisis in 2008), one may wonder whether the mechanism highlighted in this paper may have been at play in the last decade. Although regarded as an hyperinflation-related phenomenon, many authors have shown that money illusion could also be at work in a low-inflation environment. Piazzesi and Schneider (2008) developed an equilibrium model explaining the house-price booms and stock market undervaluation in both the 1970s and 2000s, which occurred in opposite interest rate and inflation regimes. Brunnermeier and Julliard (2008) also showed that reductions of inflation may lead to housing bubbles if people suffer from money illusion. Building on the work by Basak and Yan (2010), David and Veronesi (2013) developed a general equilibrium model, featuring money illusion, that captures historical stock and bond co-movements and explains the low P/E ratio and high long-term yields in the late 1970s. Finally, Miao and Xie (2013) included money illusion in a monetary model of endogenous growth, showing that the impact of money illusion on long-run growth is already significant when expected inflation is close to its long-run mean. While these contributions highlight the distortions of the equilibrium introduced by money illusion, our focus is on the individual behavior of an investor with a long-term finite horizon.

Finally, this work is related to the strand of literature attempting to explain the mispricing and liquidity puzzles of TIPS. In particular, Fleckenstein et al. (2014) found a massive and persistent mispricing of TIPS, highlighted by replicating the payoff of nominal Treasury bonds using inflation swaps and TIPS. They explained this phenomenon invoking the near-money characteristic of nominal Treasuries[3] and justified the persistent nature of the mispricing as an effect of the slow-moving-capital phenomenon. Christensen and Gillan (2017) provided additional evidence, showing that quantitative easing reduced the liquidity premium in the TIPS market. Fleming and Sporn (2013) highlighted the lack of liquidity of inflation swaps and the mispricing of inflation-related securities. The puzzling illiquidity of TIPS has led several authors to postulate the existence of a systematic liquidity factor, unique to the TIPS market. The contributions by Pflueger and Viceira (2016), Abrahams et al. (2016) and Andreasen et al. (2017) proposed several methodologies to extract this factor, finding evidence that the TIPS risk premium is time-varying and quantitatively significant.

The remainder of the paper is organized as follows. In Section 2, we set up the economic framework and derive the optimal strategy. Section 3 describes the estimation methodology, presents the dataset and discusses the parameter estimates. In Section 4, we discuss the results of the optimal portfolio strategy obtained considering constant asset risk premia, at first for a conservative investor, and then for an investor with a moderate risk aversion. In Section 5, we present additional findings obtained considering time-varying risk premia. Section 6 concludes the paper. The technical details and additional empirical findings are relegated to the Appendix.

2 Optimal portfolio choice

Our long-term investor is allowed to trade nominal and real bonds, a stock index and a nominal money market account (cash). As usual in asset allocation problems, we are in a situation of partial equilibrium, whereby the prices of the assets available for trade are given to the investor. We first

[3]See Nagel (2016) on the liquidity premium of near-money assets.

describe the economy where the agent trades, and we then derive the optimal portfolio strategy.

2.1 The economy

Stochastic discount factor (SDF) We assume that the nominal SDF dynamics is as follows:

$$\frac{\mathrm{d}\Phi_t}{\Phi_t} = -R_t \mathrm{d}t - \mathbf{\Lambda}_t' \mathrm{d}\mathbf{z}_t, \tag{1}$$

where R_t is the nominal short-term interest rate, and $\mathbf{\Lambda}_t$ is the $n \times 1$ vector of market prices of the n systematic risks $\mathbf{z}_t$.

Following the literature on dynamic term structure modeling, we assume an affine functional form for the nominal rate and for the market prices of risk:

$$R_t = R_0 + \mathbf{R}_1' \mathbf{X}_t, \tag{2}$$

$$\mathbf{\Lambda}_t = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 \mathbf{X}_t, \tag{3}$$

where R_0 is a scalar, $\mathbf{R}_1$ an $m \times 1$ vector, $\mathbf{\Lambda}_0$ an $m \times 1$ vector and $\mathbf{\Lambda}_1$ an $m \times m$ matrix; finally, $\mathbf{X}_t$ is the $m \times 1$ vector of state variables driving the dynamics of the variables of interest. We assume that these state variables are persistent and follow an autoregressive process à la Ornstein-Uhlenbeck:

$$\mathrm{d}\mathbf{X}_t = \mathbf{\Theta} \left(\bar{\mathbf{X}} - \mathbf{X}_t \right) \mathrm{d}t + \mathbf{\Sigma}_{\mathbf{X}}' \mathrm{d}\mathbf{z}_t, \tag{4}$$

where $\mathbf{\Theta}$ is a real $m \times m$ mean reversion matrix, $\bar{\mathbf{X}}$ the $m \times 1$ vector of the long-run means of the state variables and $\mathbf{\Sigma}_{\mathbf{X}}$ is the $n \times m$ volatility matrix.

Nominal quantities are converted into real quantities using the price level, for which the dynamics is assumed to be as follows:

$$\frac{\mathrm{d}P_t}{P_t} = \pi_t \mathrm{d}t + \boldsymbol{\sigma}_P' \mathrm{d}\mathbf{z}_t, \tag{5}$$

where π_t stands for the expected inflation and $\boldsymbol{\sigma}_P$ is the $n \times 1$ volatility vector of realized inflation. As for R_t and $\boldsymbol{\Lambda}_t$, we assume that expected inflation is affine in the state variables:

$$\pi_t = \pi_0 + \boldsymbol{\pi}_1' \mathbf{X}_t, \tag{6}$$

where π_0 is a scalar and $\boldsymbol{\pi}_1$ an $m \times 1$ vector.

Traded assets We assume that the long-term investor can trade a *nominally* risk-free asset (cash) yielding the short-term interest rate R_t. In addition, a stock can be traded and its price, S_t, is assumed to have the following dynamics:

$$\frac{\mathrm{d}S_t}{S_t} = \left(R_t + \boldsymbol{\sigma}_S' \Lambda_t\right) \mathrm{d}t + \boldsymbol{\sigma}_S' \mathrm{d}\mathbf{z}_t, \tag{7}$$

where $\boldsymbol{\sigma}_S$ is the $n \times 1$ volatility vector of the stock.

On the fixed income side, the investor can trade both nominal and real zero-coupon bonds. A nominal bond delivers one unit of the currency at maturity, while a real bond delivers one unit of the *numeraire*. As in Duffie and Kan (1996), nominal discount bond prices are exponentially affine functions of the state variables. The *nominal* price of a nominal discount bond with a time-to-maturity τ is given by:[4]

$$B\left(\mathbf{X}_t, \tau\right) = e^{A_0^N(\tau) + \mathbf{A}_1^N(\tau)\mathbf{X}_t}, \tag{8}$$

where the scalar $\mathbf{A}_0^N$ and the $1 \times m$ vector $\mathbf{A}_1^N$ solve the system of ODEs given in Appendix A.1. Applying Itô's lemma, we can infer the dynamics of the nominal bond price:

$$\frac{\mathrm{d}B\left(\mathbf{X}_t, P_t, T-t\right)}{B\left(\mathbf{X}_t, P_t, T-t\right)} = \left(R_t + \mathbf{A}_1^N \boldsymbol{\Sigma}_\mathbf{X}' \boldsymbol{\Lambda}_t\right) \mathrm{d}t + \mathbf{A}_1^N \boldsymbol{\Sigma}_\mathbf{X}' \mathrm{d}\mathbf{z}_t. \tag{9}$$

[4]See section A.1 in the Appendix.

Similarly, the *nominal* price of a real zero-coupon bond is given by:[5]

$$I\left(\mathbf{X}_t, P_t, \tau\right) = P_t e^{A_0^I(\tau)+\mathbf{A}_1^I(\tau)\mathbf{X}_t}, \tag{10}$$

and its dynamics by:

$$\frac{\mathrm{d}I\left(\mathbf{X}_t, P_t, T-t\right)}{I\left(\mathbf{X}_t, P_t, T-t\right)} = \left(R_t + \left(\mathbf{A}_1^I \boldsymbol{\Sigma}'_{\mathbf{X}} + \boldsymbol{\sigma}'_P\right)\boldsymbol{\Lambda}_t\right)\mathrm{d}t + \left(\mathbf{A}_1^I \boldsymbol{\Sigma}'_{\mathbf{X}} + \boldsymbol{\sigma}'_P\right)\mathrm{d}\mathbf{z}_t. \tag{11}$$

As can be noticed, not only real bond prices span the innovations in the state variables $\mathbf{X}_t$, but also realized inflation P_t.

To simplify the notation, we denote the *nominal* price of a generic risky asset at time t by Y_t^i. Its dynamics takes the following form:

$$\frac{\mathrm{d}Y_t^i}{Y_t^i} = \left(R_t + \boldsymbol{\sigma}'_{Y^i}\boldsymbol{\Lambda}_t\right)\mathrm{d}t + \boldsymbol{\sigma}'_{Y^i}\mathrm{d}\mathbf{z}_t. \tag{12}$$

Preferences Consider a long-term investor endowed with utility from real terminal wealth:

$$U\left(w_T\right) = \frac{w_T^{1-\gamma}}{1-\gamma}, \tag{13}$$

where w_T stands for the real wealth at the investor's horizon T. When the investor is money-illusioned, we assume that her objective function modifies as follows:

$$U\left(w_T\right) = \frac{\left(w_T^{1-\alpha}W_T^{\alpha}\right)^{1-\gamma}}{1-\gamma}, \tag{14}$$

where W_T stands for the nominal terminal wealth and $0 \leq \alpha \leq 1$ measures the degree of money illusion. When $\alpha = 0$, the investor is rational, in the sense that she maximizes her expected utility

[5] See section A.2 in the Appendix.

from real terminal wealth. When $\alpha = 1$, the investor is completely money-illusioned, as she reasons only in nominal terms. This specification is inspired by Basak and Yan (2007), Miao and Xie (2013)and David and Veronesi (2013), although we choose to define utility over terminal wealth for analytical tractability. In particular, when markets are complete, a quasi-closed-form solution can also be derived for the case of utility over consumption. However, a specification with utility over terminal wealth also allows us to obtain analytical results in some relevant cases of suboptimal portfolio strategies and market incompleteness. These include when the investor follows a strategy with a different value of α from the value considered as rational, or when the investment universe includes fewer non-redundant assets than the number of sources of risk in the market.

Given that the relationship between real and nominal wealth is given by $w_T = W_T P_T^{-1}$, the objective function becomes:

$$U\left(w_T\right) = \frac{\left(w_T^{1-\alpha} W_T^{\alpha}\right)^{1-\gamma}}{1-\gamma} = \frac{W_T^{1-\gamma} P_T^{-(1-\alpha)(1-\gamma)}}{1-\gamma} \equiv U\left(W_T\right), \tag{15}$$

with the understanding that the rational case is nested by setting $\alpha = 0$.

Budget constraint Investors allocate their wealth to N risky assets and the money market account. The dynamics of nominal wealth reads as follows:

$$\frac{\mathrm{d}W_t}{W_t} = \sum_{i=1}^{N} \omega_t^i \frac{\mathrm{d}Y_t^i}{Y_t^i} + \left(1 - \sum_{i=1}^{N} \omega_t^i\right) R_t \mathrm{d}t, \tag{16}$$

where ω_t^i stands for the proportion of (indifferently) real or nominal wealth invested in risky asset i. Using (12), this dynamics can be written as follows:

$$\frac{\mathrm{d}W_t}{W_t} = \left(R_t + \boldsymbol{\omega}_t' \boldsymbol{\Sigma}' \boldsymbol{\Lambda}_t\right) \mathrm{d}t + \boldsymbol{\omega}_t' \boldsymbol{\Sigma}' \mathrm{d}\mathbf{z}_t, \tag{17}$$

where $\boldsymbol{\omega}_t$ is the $N \times 1$ vector of weights and $\boldsymbol{\Sigma}$ is a matrix which columns are the volatility vectors of the risky assets, $\boldsymbol{\sigma}_{Y^i}$. The dynamics of real wealth is obtained by applying Itô's lemma to $w_t = W_t P_t^{-1}$:

$$\frac{\mathrm{d}w_t}{w_t} = \left(R_t - \pi_t + \boldsymbol{\omega}_t' \boldsymbol{\Sigma}' \left(\boldsymbol{\Lambda}_t - \boldsymbol{\sigma}_P\right) + \boldsymbol{\sigma}_P' \boldsymbol{\sigma}_P\right) \mathrm{d}t + \left(\boldsymbol{\omega}_t' \boldsymbol{\Sigma}' - \boldsymbol{\sigma}_P'\right) \mathrm{d}\mathbf{z}_t.$$

Market completeness Concerning the estimation of the model, we will see in the next section that the market is complete, as it is possible to pin down the market prices for all risks introduced in the economy, and therefore to fully characterize the dynamics of the SDF (1). In order to derive the optimal portfolio strategy, however, it is important to take into account the issue of market completeness from the point of view of the investor. The market is complete if there are at least $N = n$ non-redundant assets available for trade. As we will consider $n = 5$, we could meet this conditions, but, in order to obtain results more easily interpretable from the economic point of view, we prefer to study the optimal allocation considering at most one nominal bond, one inflation-indexed bond and the stock market, on top of the nominal risk-less asset.[6] When utility depends only on terminal wealth, as is well known since Kim and Omberg (1996), it is possible to solve the optimal allocation problem even when markets are not complete. In this case, knowing the dynamics of traded assets does not allow investors to completely span the dynamics of the SDF. To solve this issue, along the lines of He and Pearson (1991) and Sangvinatsos and Wachter (2005), we write the market prices of risk $\boldsymbol{\Lambda}_t$ as the sum of two components: the first, $\boldsymbol{\Lambda}_t^*$, corresponding to their projection onto the returns of the assets available for trade; and the second, $\boldsymbol{\nu}_t$, orthogonal to the traded assets. In principle, there exists an infinity of plausible vectors $\boldsymbol{\nu}_t$. We show in Appendix B.1 how, among all the possible values of $\boldsymbol{\nu}_t$, it is possible to impose that the dynamics of optimal wealth is actually spanned by the traded assets, pinning down the unique vector $\boldsymbol{\nu}_t^*$ that makes the

[6] When several highly correlated bonds are available for trade, optimal unconstrained portfolio positions may reach unrealistically high values, as shown by Sangvinatsos and Wachter (2005).

optimal wealth achievable with the traded assets. The nominal SDF of the investor is then:

$$\frac{\mathrm{d}\Phi_t^{\boldsymbol{\nu}^*}}{\Phi_t^{\boldsymbol{\nu}^*}} = -R_t \mathrm{d}t - \left(\boldsymbol{\Lambda}_t^* + \boldsymbol{\nu}_t^*\right)' \mathrm{d}\mathbf{z}_t. \tag{18}$$

Equivalently, it is possible to define a real SDF, denoted as $\phi_t^{\boldsymbol{\nu}^*} \equiv P_t \Phi_t^{\boldsymbol{\nu}^*}$.

2.2 Optimal portfolio choice under money illusion

Our setting allows us to write a separable value function for the investor's problem:

$$J\left(W_t, t\right) \equiv \max_{\left[\boldsymbol{\omega}_s\right]_{s=t}^T} \mathrm{E}_t \left[\frac{W_T^{1-\gamma} P_T^{-(1-\alpha)(1-\gamma)}}{1-\gamma}\right] = \frac{W_t^{1-\gamma} P_t^{-(1-\alpha)(1-\gamma)}}{1-\gamma} \left[F\left(\mathbf{X}_t, t, T\right)\right]^{\gamma}. \tag{19}$$

The optimal strategy uncovers the typical structure à la Merton:

$$\boldsymbol{\omega}_t = \frac{1}{\gamma} \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1} \boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t - (1-\alpha) \frac{1-\gamma}{\gamma} \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1} \boldsymbol{\Sigma}'\boldsymbol{\sigma}_P + \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1} \boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}} \frac{\left(F_{\mathbf{X}}\right)'}{F}, \tag{20}$$

where $F_{\mathbf{X}}$ is the column vector of the partial derivatives of F. We show in Appendix B.1 that F takes the form:

$$F\left(\mathbf{X}_t, \tau\right) = \exp\left\{\frac{1}{2}\mathbf{X}_t'\mathbf{B}_3\left(\tau\right)\mathbf{X}_t + \mathbf{B}_2\left(\tau\right)\mathbf{X}_t + B_1\left(\tau\right)\right\},$$

where $\mathbf{B}_3\left(\tau\right)$, $\mathbf{B}_2\left(\tau\right)$ and $B_1\left(\tau\right)$ are the solution of a system of Riccati equations.

Taking $\tilde{\mathbf{B}}_3\left(\tau\right) = \frac{\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)}{2}$, we can rewrite the optimal portfolio strategy (20) as:

$$\begin{aligned} \boldsymbol{\omega}_t =& \frac{1}{\gamma} \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1} \boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t - (1-\alpha) \frac{1-\gamma}{\gamma} \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1} \boldsymbol{\Sigma}'\boldsymbol{\sigma}_P \\ &+ \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1} \boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}} \left(\tilde{\mathbf{B}}_3\left(\tau\right)\mathbf{X}_t + \mathbf{B}_2'\left(\tau\right)\right). \end{aligned} \tag{21}$$

The first component is the mean-variance speculative component, taking advantage of the instantaneous risk/return trade-off offered by the assets available for trade. The second component hedges instantaneous realized inflation risk and elicits a direct impact of money illusion. This second term disappears under severe money illusion ($\alpha = 1$), which means that the investor is no longer concerned by realized inflation risk. Risk attached to future inflation does matter for intertemporal hedging purposes. The last term is the intertemporal hedging component, which is affected by money illusion in a non-linear way. The portfolio strategy is still linear in the state variables ($\mathbf{X}$), driving the macroeconomic variables (short rate and expected inflation) and the market prices of risk.

To better gather the economics behind the above strategy (20), note that it can be rewritten as:

$$\begin{aligned}\boldsymbol{\omega}_t =& \frac{1}{\gamma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\left[\boldsymbol{\Lambda}_t - (1-\alpha)\,\boldsymbol{\sigma}_P\right] + (1-\alpha)\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P \\ &+ \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\tilde{\mathbf{B}}_3(\tau)\,\mathbf{X}_t + \mathbf{B}_2'(\tau)\right).\end{aligned} \tag{22}$$

The investor is interested in the real risk/return trade-off, i.e. the real risk premia traded assets can offer. As such, for a non-illusioned investor, the first mean-variance component involves the *real* market prices of risk $\boldsymbol{\Lambda}_t - \boldsymbol{\sigma}_P$, which do indeed represent the volatility of the real SDF. However, in the presence of money illusion, the investor accounts to a lesser extent for realized inflation volatility, even ignoring it under severe money illusion ($\alpha = 1$). The same happens to the second term, which does not depend on risk aversion: a non-illusioned investor should try to hedge (perfectly or imperfectly) the inflation risk exposure, but, once again, money illusion distorts this behavior and leads a perfectly illusioned investor to ignore unexpected inflation risk. Overall, money illusion affects the perception of the risk/return trade-off the investor is subject to, as well as the unexpected inflation risk to hedge.

What about the intertemporal hedging component? As is well known, this component brings the horizon effects into the strategy and it is interesting to assess the potential impact of money illusion on this component. For this purpose, we start looking at the case of an extremely conservative investor, that is, an investor with an infinite risk aversion. We show in Appendix B.2 that, when the market prices of risk are constant ($\mathbf{\Lambda}_1 = \mathbf{0}$), the optimal strategy is:

$$\boldsymbol{\omega}_t^{\gamma\to\infty} = (1-\alpha)\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\left(\mathbf{\Sigma}_{\mathbf{X}}\mathbf{A}_1^{I}(\tau) + \boldsymbol{\sigma}_P\right) + \alpha\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\mathbf{\Sigma}_{\mathbf{X}}\mathbf{A}_1^{N}(\tau)\,. \tag{23}$$

Remembering that the volatility vector of an indexed bond is $\mathbf{\Sigma}_{\mathbf{X}}\mathbf{A}_1^{I}(\tau) + \boldsymbol{\sigma}_P$ and that the volatility vector of a nominal bond is $\mathbf{\Sigma}_{\mathbf{X}}\mathbf{A}_1^{N}(\tau)$, it is clear that a non-illusioned investor ($\alpha = 0$) avoids speculating through the risky assets (bonds and stocks) and invests all the wealth in an indexed bond, which maturity coincides with the investment horizon. A money-illusioned investor, conversely, combines an indexed and a nominal bond, both maturing at her investment horizon. The relative weight of the two is related to the degree of money illusion. An extremely illusioned investor ($\alpha = 1$) invests only in a nominal bond with the appropriate maturity.

From the strategy in (23), it appears that realized inflation is ignored by severely money-illusioned investors. What about expected inflation? As shown in Appendix B.2, the optimal strategy for a conservative investor can also be written as:

$$\begin{aligned}\boldsymbol{\omega}_t^{\gamma\to\infty} \quad = \quad & (1-\alpha)\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\boldsymbol{\sigma}_P + \left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\mathbf{\Sigma}_{\mathbf{X}}\left[\mathbf{R}_1'\left(\mathrm{e}^{-\mathbf{\Theta}\tau} - \mathbf{I}\right)\mathbf{\Theta}^{-1}\right]' \\ & -(1-\alpha)\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\mathbf{\Sigma}_{\mathbf{X}}\left[\boldsymbol{\pi}_1'\left(\mathrm{e}^{-\mathbf{\Theta}\tau} - \mathbf{I}\right)\mathbf{\Theta}^{-1}\right]'.\end{aligned} \tag{24}$$

It appears that non-illusioned investors, other than hedging unexpected inflation, aim to hedge the real rate, that is, the difference between the nominal rate (loading on the state variables with the vector $\mathbf{R}_1$) and expected inflation (loading on the state variables with the vector $\boldsymbol{\pi}_1$). Conversely, money-illusioned investors focus on hedging the short-term nominal rate only, since the two components related to realized and expected inflation vanish for $\alpha = 1$. Clearly, if expected inflation affects the nominal short-term rate (for example through a monetary policy rule), the illusioned investor indirectly still partially hedges expected inflation, but expected inflation is not a source of risk which matters per se.

When risk aversion is finite and risk premia are allowed to be time-varying, there is no immediately interpretable explicit solution. However, it is worth remembering that:

$$F\left(\mathbf{X}_t, t, T\right) = E_t\left[\left(\frac{\Phi_T^{\boldsymbol{\nu}^*}}{\Phi_t^{\boldsymbol{\nu}^*}}\left(\frac{P_T}{P_t}\right)^{1-\alpha}\right)^{1-\frac{1}{\gamma}}\right]. \tag{25}$$

A non-illusioned investor focuses on the quantities driving the real pricing kernel $\phi_t^{\boldsymbol{\nu}^*} = \Phi_t^{\boldsymbol{\nu}^*} P_t$, which are the real rate, unexpected inflation and the market prices of risk. A severely money-illusioned investor instead focuses on the risks driving the nominal pricing kernel, which are the nominal short-term interest rate and the market prices of risk. Expected and realized inflation are not relevant to a money-illusioned investor. In the empirical analysis, we assess the quantitative importance of these effects.

3 Estimation

In this section, we estimate the model. After describing the dataset used, we present the methodology employed, highlighting the characteristics of the different specifications for the risk premia that we consider. Finally, we discuss the estimates of the parameters.

3.1 Dataset

We estimate the model using U.S. monthly data from 31st January 1999 until 31st January 2016. We consider zero-coupon nominal yields for the following maturities: 3 and 6 months, and 1, 2, 3, 5, 7 and 10 years. The 3- and 6-month yields were obtained from the Treasury Bills rates, available on the Federal Reserve Economic Data website[7] (series GS3M and GS6M). The other nominal zero-coupon yields are the series fitted by Gürkaynak et al. (2007), available on the website of the Federal Reserve Board.[8] We use zero-coupon real yields for the maturities of 5, 7 and 10 years, as fitted in Gürkaynak

[7] `https://research.stlouisfed.org/fred2/`

[8] `https://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html`

et al. (2010), which are also available on the website of the Federal Reserve Board.[9] As a broad U.S. stock market index, we consider the CRSP NYSE/Amex/NASDAQ/ARCA Value-Weighted Market Index, extracting the end-of-month data from the daily series. To compute realized inflation, we use the Consumer Price Index for All Urban Consumers: All Items (CPIAUCSL), available at a monthly frequency on the Federal Reserve Economic Data website.

3.2 Methodology

Along the lines of Joslin et al. (2011), we estimate the model by maximum likelihood, by choosing as pricing factors the first three principal components of the whole set of the time series of observed nominal and real yields, which we collect in the vector of state variables $\mathbf{X}_t$.[10] We stack the state variables $\mathbf{X}_t$, the log price index $\log P_t$ and the log stock index level $\log S_t$ into a column vector $\mathbf{Z}_t$:

$$\mathbf{Z}_t = \begin{bmatrix} X_t^1 & X_t^2 & X_t^3 & \log(S_t) & \log(P_t) \end{bmatrix}', \tag{26}$$

for which the dynamics is:

$$\mathrm{d}\mathbf{Z}_t = \mathbf{B}\mathrm{d}t + \mathbf{A}\mathbf{Z}_t\mathrm{d}t + \mathbf{\Sigma}'_{\mathbf{Z}}\mathrm{d}\mathbf{z}_t, \tag{27}$$

where the column vector $\mathbf{B}$ and the matrix $\mathbf{A}$ can be compactly written as:

$$\mathbf{B} = \begin{bmatrix} \mathbf{\Theta}\bar{\mathbf{X}} \\ R_0 + \boldsymbol{\sigma}'_S\mathbf{\Lambda}_0 - \frac{\|\boldsymbol{\sigma}_S\|^2}{2} \\ \pi_0 - \frac{\|\boldsymbol{\sigma}_P\|^2}{2} \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} & 0 & 0 \\ -\mathbf{\Theta} & 0 & 0 \\ & 0 & 0 \\ \mathbf{R}'_1 + \boldsymbol{\sigma}'_S\mathbf{\Lambda}_1 & 0 & 0 \\ \boldsymbol{\pi}'_1 & 0 & 0 \end{bmatrix}, \tag{28}$$

[9] https://www.federalreserve.gov/pubs/feds/2008/200805/200805abs.html

[10] We also tried considering two separate level factors for the nominal and real yields, as in Christensen et al. (2010), but we did not notice any significant difference in the quality of the estimates, nor in the implications in terms of portfolio choice. This is probably because of the fact that, differently from their work, we do not impose any constraints on the matrix $\mathbf{K}_1$, as we are not interested in obtaining a representation à la Nelson-Siegel of the term structure.

where we set $\bar{\mathbf{X}} = \mathbf{0}$, as the state variables are the principal components of bond yields and are centered around zero. The volatility matrix $\mathbf{\Sigma_Z}$ is obtained by juxtaposing the matrix $\mathbf{\Sigma_X}$ and the vectors $\boldsymbol{\sigma}_S$ and $\boldsymbol{\sigma}_P$:

$$\mathbf{\Sigma_X} = \begin{bmatrix} \Sigma_{\mathbf{X}}(1,1) & \Sigma_{\mathbf{X}}(1,2) & \Sigma_{\mathbf{X}}(1,3) \\ 0 & \Sigma_{\mathbf{X}}(2,2) & \Sigma_{\mathbf{X}}(2,3) \\ 0 & 0 & \Sigma_{\mathbf{X}}(3,3) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_P = \begin{bmatrix} \sigma_P(1) \\ \sigma_P(2) \\ \sigma_P(3) \\ \sigma_P(4) \\ 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_S = \begin{bmatrix} \sigma_S(1) \\ \sigma_S(2) \\ \sigma_S(3) \\ \sigma_S(4) \\ \sigma_S(5) \end{bmatrix}.$$

Applying a Euler scheme, we perform an exact discretization of this joint continuous-time process, which constitutes the first contribution to the log-likelihood function.[11] The second contribution to the likelihood function is obtained by imposing the bond pricing restrictions, which relate the current value of the state variables $\mathbf{X}_t$ to the observed nominal and real bond yields. We allow for Gaussian observation errors, uncorrelated both in time series and cross-sectionally, with a constant standard deviation σ_ε^B for the nominal yields and σ_ε^I for the real yields.

We numerically maximize the likelihood function with respect to the whole set of model parameters at the same time, by considering different alternatives to the restrictions that can be imposed on the time-varying market prices of risk. Firstly, we consider a specification with the restriction that the asset risk premia are constant, which we obtain imposing that $\mathbf{\Lambda}_1 = \mathbf{0}$. Secondly, we consider a specification with time-varying risk premia, initially with no restrictions imposed on the matrix $\mathbf{\Lambda}_1$. As in Christensen et al. (2010), we then iterate the estimation, by progressively imposing a zero restriction on the element of $\mathbf{\Lambda}_1$ with the lowest t-statistics, stopping when all the elements of $\mathbf{\Lambda}_1$ have a t-stat higher than 2.[12] Thirdly, we consider a specification where we let the risk premia of the risky assets vary, but constraining their volatility to some reasonable values. In particular, we

[11] For details on the exact discretization of the continuous-time process and on the construction of the likelihood function, refer to Sangvinatsos and Wachter (2005) or Koijen et al. (2010).

[12] Similar procedures have been followed by several authors, such as Duffee (2002), Sangvinatsos and Wachter (2005) and Christensen et al. (2010). Joslin et al. (2014) performed a model selection among all possible sets of zero restrictions on the elements of $\mathbf{\Lambda}_1$. This would entail very long computation times in our continuous-time framework.

impose that the volatility of the risk premia of the nominal bonds, $\left\| \mathbf{A}_1^N \mathbf{\Sigma}'_{\mathbf{X}} \mathbf{\Lambda}_1 \mathbf{\Sigma}'_{\mathbf{X}} \right\|$, and the real bonds, $\left\| \left(\mathbf{A}_1^I \mathbf{\Sigma}'_{\mathbf{X}} + \boldsymbol{\sigma}'_P \right) \mathbf{\Lambda}_1 \mathbf{\Sigma}'_{\mathbf{X}} \right\|$, are not higher than the volatility of the short-term interest rate, which is equal to 0.64% per annum.[13] We also impose that the volatility of the risk premium of realized inflation, $\|\boldsymbol{\sigma}'_P \mathbf{\Lambda}_1 \mathbf{\Sigma}'_{\mathbf{X}}\|$, is not higher than 0.5% per annum and, finally, that the volatility of the equity premium, $\|\boldsymbol{\sigma}'_S \mathbf{\Lambda}_1 \mathbf{\Sigma}'_{\mathbf{X}}\|$, is not higher than 1% per annum. By imposing economically reasonable restrictions on the time variation of risk premia, this methodology attempts to implement, as suggested by Sarno et al. (2016), a modeling approach that is flexible but limits overfitting.

For the empirical study, we consider as base case the results obtained for constant risk premia, which is for our purpose the most reliable framework, allowing us to focus on the roles of interest rate, expected inflation and realized inflation hedging. We also present the results obtained for the case of volatility-constrained time-varying risk premia. We relegate to the Appendix the results obtained in the case where the statistically significant elements of $\mathbf{\Lambda}_1$ are left unconstrained, justifying why we deem that this framework is not appropriate for the analysis.

3.3 Parameter estimates

Table 1 shows the parameter estimates for the specification with constant risk premia (Panel (a)) and with volatility-constrained risk premia (Panel (b)). The estimates of the parameters related to the instantaneous nominal risk-free rate, R_0 and $\mathbf{R}_1$, are, as expected, almost identical in the two settings, as well as the volatility vectors and the vector of constant risk premia $\mathbf{\Lambda}_0$. R_0 is very close to the average 3-month nominal yield (1.85% vs 1.87%). π_0 represents the drift of the price index under the historical probability measure and is similar between the two settings, being equal to about 2.10%. The average of the instantaneous real rate is given by $R_0 - \pi_0 + \boldsymbol{\sigma}'_P \mathbf{\Lambda}_0 = 0.92\%$. The vector of loadings $\boldsymbol{\pi}_1$ is different between the two settings, but we verified that, as expected, the corresponding quantities under the pricing measure, $\boldsymbol{\pi}_1 - \mathbf{\Lambda}'_1 \boldsymbol{\sigma}_P$, are almost identical. The same applies to the mean-reversion matrices, $\mathbf{\Theta}$, which are different between the two settings, but

[13] We choose to impose this restriction for the 3- and 10-year nominal bonds, and for the 7-year real bond.

the quantities $\mathbf{\Theta} + \mathbf{\Sigma}'_{\mathbf{X}}\mathbf{\Lambda}_1$ are equal to each other. Finally, the standard deviations of the pricing errors relative to the nominal yields, σ^B_ϵ, are in both settings equal to 12 basis points, while the corresponding quantities for real yields, σ^I_ϵ, are both equal to 7 basis points.

[Table 1 about here.]

The goodness of fit relative to the historical distributions can be checked in Table 2, where we report the annualized mean values and the volatilities, both historical and model-implied, of bond yields, realized inflation and realized equity returns. The two specifications fit the historical moments very well. The model-implied means of the risk premia are also very similar in the two settings. The fitted risk premia of the nominal bonds are about $0.3\% - 0.5\%$ higher than the risk premia of inflation-indexed bonds for the same maturities. The realized inflation risk premium is definitely non-negligible and about 1.2% in both settings, while the equity premium is just below 5%. The risk premia volatilities for the second setting are close to the bounds imposed, i.e. the bond premia volatilities are close to the volatility of the historical 3-month rate (0.64%), while the realized inflation and equity risk premium volatilities are respectively 0.5% and 1%. Finally, the short-rate volatilities are similar between the two settings, while the model-implied volatility of the expected inflation is slightly higher when the risk premia are time-varying.

[Table 2 about here.]

Table 3 shows, whenever available, the pairwise correlations between returns and economic variables, both from the historical distribution and as implied by the estimated parameters for the two specifications. The model-implied pairwise correlations are rather similar to each other and fit the historical values reasonably well, with some exceptions among the correlations involving short-term nominal yields. Real bond returns, differing from nominal bond returns, are positively correlated with the price index. They also have a weak correlation with equity returns, while nominal bond returns are negatively correlated with the equity. Nominal and real bond returns corresponding to the same maturities tend to be strongly positively correlated. Furthermore, it is interesting to look at the model-implied correlations between asset returns and the unobservable economic variables. Nominal bond returns seem to be more (negatively) correlated with the short-term rate R w.r.t. real bond returns for the same maturities. Real bond returns, differing from nominal bond returns, are strongly positively correlated with the innovations in the expected inflation π and strongly negatively correlated with the real rate r.

[Table 3 about here.]

The first row of graphs in Figure 2 shows, for the two specifications, the model-implied short-term interest rate, the expected inflation, the break-even inflation and the Blue Chip inflation forecast. The expected inflation is slightly higher than the break-even inflation, and the difference between the two is the realized inflation risk premium. The expected inflation is overall in line with the Blue Chip forecast in both cases. The second row of graphs represents the risk premia of a 10-year nominal bond, a 10-year real bond, the stock and the realized inflation. In the model with time-varying volatility-constrained premia, these are centered around the values for the constant risk premia specification, and their ranges of variation seem to be reasonable, going from about 2% to 5% for the two 10-year bonds, from -2% to 9% for the stock and from about 0% to 2% for the realized inflation premium. The third row of graphs show the myopic allocation followed by a mean-variance investor with a risk aversion $\gamma = 10$. When risk premia are time-varying, these positions also vary with time. As can be noticed, the positions range from -50% to about 100%, without reaching excessive levels of leverage or short selling, as expected by an investor with a moderate level of risk aversion. The fourth row shows the maximum ex-ante Sharpe ratio achievable with the three assets above, which is about 0.6 for the specification with constant risk premia, and ranges between 0.4 and 1 for the model with time-varying risk premia.

[Figure 2 about here.]

Figure A.3 in Appendix D.1 shows the same time series as Figure 2 for the specification with time-varying risk premia and no volatility constraints. The time series of risk premia, the mean-variance portfolio weights and the maximum achievable Sharpe ratio are subject to variations which are unreasonably large.[14] These quantities are indeed not directly observable, and a model that over-fits in-sample data has the drawback of returning uncontrollably volatile time series for these unobservable quantities.

For our empirical analysis, we choose to use as base case the model with constant risk premia and to verify that the results are consistent with those obtained considering the specification with volatility-constrained risk premia. We discuss in Appendix D why, although qualitatively confirming most of our empirical results, we do not deem the specification with volatility-unconstrained risk premia to be reliable for our analysis. Our choice is consistent with the findings of Feldhütter et al. (2012) and Sarno et al. (2016), who argued that (unconstrained) essentially affine term structure models are very sensitive to estimation errors and in-sample overfitting. We also take into account the in-sample empirical results of Sangvinatsos and Wachter (2005) and Barillas (2011), who found very large portfolio positions and unrealistically high utility losses associated with suboptimal portfolio strategies.[15]

4 Main empirical findings

We start our empirical investigation considering the case where risk premia are constant. We have already argued that this is a robust setting in an asset allocation context, being less prone to over-fitting than models with time-varying premia and, as pointed out by Feldhütter et al. (2012), less sensitive to estimation errors. Furthermore, in this context, we can unambiguously associate the

[14] Duffee (2011) noticed the same empirical fact and attempted to reduce the issue of estimation overfitting by imposing a numerical constraint on the average value of the maximum achievable Sharpe ratio.

[15] To the best of our knowledge, only Duffee (2002) documented an acceptable out-of-sample predictive ability of unconstrained essentially affine term structure models. This analysis, however, is based on rather small sample (less than 4 years of out-of-sample monthly observations). Duffee (2002) also found a suspiciously high in-sample model-implied volatility of bond risk premia.

intertemporal hedging demands, and their variations corresponding to different degrees of money illusion, to the economic variables, such as the short-term rate, the expected inflation and the real rate. We first consider a conservative investor, that is, an individual with an infinite risk aversion, and we then extend the analysis to the case of a moderate investor, that is, an individual with a medium level of risk aversion ($\gamma = 10$).

4.1 Conservative investor

Portfolio strategy In the case of $\gamma \to \infty$ and constant risk premia it is possible to explicitly determine the optimal portfolio strategy, as in (23). Figure 3 reports the positions in the four assets for different degrees of money illusion. $\alpha = 0$ corresponds to a non-illusioned investor, $\alpha = 0.5$ to a partially illusioned investor and $\alpha = 1$ to a totally money-illusioned investor. The state variables are at their long-run means.

[Figure 3 about here.]

A conservative investor with a very short horizon invests only in the money market account (cash). Increasing the horizon leads to an investment in the other assets. A particular case arises when the horizon equals 10 years, that is the maturity of the bonds available for trade. A rational investor ($\alpha = 0$) allocates all wealth in the indexed bond and nothing in the other assets. For intermediate levels of money illusion, the investor reduces the position in the real bond and increases the position in the nominal bond. A severely money-illusioned investor ($\alpha = 1$) invests only in the nominal bond. For any investment horizon, the stock and the cash positions are barely affected by the degree of money illusion.

In terms of intertemporal hedging demands, an interesting pattern appears: while the position in the nominal bond flattens when the investment horizon is above 10 years, the position in the real bond keeps increasing steadily. Roughly, a rational investor allocates 0% in the nominal bond if the investment horizon is 10 years and about 20% when the horizon is 20 or 30 years. The corresponding figures for the position in the indexed bond are 100%, 150% and 170%, respectively. This is a direct consequence of the dynamics of expected inflation, which loads more than the nominal short-term rate on the less persistent state variables driving the economy (X_2 and X_3).

The stock position for a conservative investor is very small, it is almost not sensitive to money illusion and, as expected, is increasing in the investment horizon. It is equal to zero when the latter is equal to 10 years, i.e. the maturity of the bonds. The cash position decreases from 100% for short horizons to zero when the horizon is equal to 10 years, when the portfolio is fully invested in bonds. The cash position keeps decreasing toward negative values, implying leveraged positions in the other assets, for horizons longer than 10 years.

While money illusion strongly shifts the portfolio from indexed to nominal bonds, the indexed bond position is zero only when the nominal bond maturity is equal to the investment horizon and money illusion is severe. Overall, given that nominal bonds for almost any maturity up to 30 years exist, we can confidently say that money illusion drives conservative investors away from the indexed bond market, as a conservative money-illusioned investor allocates all wealth into a nominal bond maturing at her investment horizon. Conversely, if indexed bonds were available for all maturities, conservative non-illusioned investors would invest only in indexed bonds.

Intertemporal hedging demands It is evident that the intertemporal hedging components (horizon effects) of the optimal portfolio strategy in Figure 3 are substantial, and money illusion causes a shift from the indexed to the nominal bond market. Given that market prices of risk are constant so far, is this related to expected inflation or the nominal short-term rate? In Figure 4, we show the projections of the intertemporal hedging components of each asset position onto the expected inflation and the nominal short-term rate, as well as the corresponding orthogonal components.

[Figure 4 about here.]

As can be noticed, the component of intertemporal hedging projected onto expected inflation is substantial and very sensitive to the degree of money illusion, while the orthogonal component is far less sensitive. The mechanism at play is straightforward: investors take long positions on indexed bonds and short nominal bonds in order to hedge expected inflation. Money illusion tends to decrease (in absolute value) these positions. The intertemporal hedging demand corresponding to the stock is mostly correlated to expected inflation and, as stock returns are positively correlated with the expected inflation π (see Table 3b), the hedging demand is decreasing with α. However, as the inflation-indexed bond has better inflation-hedging properties, the stock position is very small.

The two bottom rows of Figure 4 deliver a totally different picture, as the intertemporal hedging activity projected onto the nominal short-term rate is negligible, and the sensitivity to α is also very small. One may be surprised that, even for a money-illusioned investor, the projection of the hedging demand onto the nominal risk-free rate is small. This is due to the low correlation between the short-term rate and the 10-year nominal and real bonds, as documented in Table 3.

In short, when risk premia are constant, the intertemporal hedging activity for a conservative investor is linked to the rationale of hedging expected inflation, while the nominal interest rate plays a marginal role. Money illusion tends to reduce the demand for assets hedging dynamic variations of expected inflation. When both types of bonds are available, they are used to perform most of the intertemporal hedging activity, while the stock plays a marginal role.

Impact of money illusion on equity investments In order to further investigate the impact of money illusion on stock positions, we show in Figure 5 the portfolio strategy of rational and illusioned investors when one of the bonds (either nominal or real) is removed from the asset universe.

[Figure 5 about here.]

Figure 5a represents a situation where the investor does not have access to the inflation-indexed bond market, which is nowadays a realistic situation in many financial markets. The nominal bond position is increasing in the investment horizon, while we saw in Figure 3 that it flattens when indexed bonds are available, and is essentially insensitive to money illusion. The stock position tends to increase with the horizon and is substantial (respectively 10% and 20% for a 10- and a 20-year horizon) for a non-illusioned investor, as the stock is used to hedge expected inflation in spite of the real bond. The stock position is instead substantially reduced when the degree of money illusion is higher.

Figure 5b shows that, when only indexed bonds are available (which is not a realistic situation in current fixed-income markets), their position is monotonically increasing in the investment horizon and decreases with the degree of money illusion. Furthermore, this long position is partially offset by a negative stock position when the degree of money illusion is high, while the stock position is virtually zero for a non-illusioned investor, who benefits from the inflation-hedging properties of the real bond.

Synthesis A conservative money-illusioned investor shuns the stock market and takes small positions in the indexed bond. The impact of money illusion is substantial, as it drives investors out of the indexed bond market toward the nominal bond market.

4.2 Moderate investor

Consider now a moderate investor, with risk aversion of $\gamma = 10$. For this investor, the risk/return trade-off offered by the opportunity set plays a role also through the speculative component, which is represented by the first term in (22). We want to assess the impact of money illusion in this context and see how the optimal portfolio strategy is similar and how it differs from the results obtained for the infinitely risk-averse investor. Furthermore, as risk aversion is finite, we are able to calculate an annualized certainty equivalent for the investment and to make welfare considerations.

Portfolio strategy The portfolio strategy shown in Figure 6 mimics the pattern observed for the conservative investor, although with respect to the case of $\gamma \to \infty$, the optimal positions are offset, as the first speculative component of (22) is nonzero. A rational investor with a 10-year horizon invests 60% in the nominal bond, rather than 0% as the conservative investor. Conversely, the position in the indexed bond is around 80%, rather than 100%. In spite of these differences, money illusion affects the portfolio strategy similarly to the case of a conservative investor, as it entails a reduction of the optimal position in the real bond and an increase of the position in the nominal bond. Furthermore, the amount of this substitution effect is similar to that observed in Figure 3. As expected, the stock position is higher compared to the case of a conservative investor, but the optimal position is again almost insensitive to money illusion.

[Figure 6 about here.]

Utility loss due to money illusion Of particular interest is the welfare cost of money illusion. We estimate it considering the portfolio strategy followed by an agent with a degree of money illusion α and calculate the expected utility perceived by a non-illusioned investor forced to follow that strategy. The cost is expressed in terms of annualized certainty equivalent loss. The derivation of the value function for an investor following a strategy suboptimal for her preferences is detailed in Appendix B.3, while in Appendix B.4 we describe how we compute the loss.

The opportunity cost ℓ_{ann}, for different degrees of money illusion α, is shown in the bottom graph of Figure 6. As expected, the annualized loss increases with α, but not linearly, as for a 10-year horizon and $\alpha = 0.5$, we observe an annualized loss of 0.25%, while for a totally illusioned agent ($\alpha = 1$), the loss is almost 1% per annum. Furthermore, the annualized loss steeply increases with the investment horizon up to around 10 years, and keeps increasing at a lower rate when the horizon is longer. For $\alpha = 1$ and a 30-year horizon, the annualized loss is about 1.2%.

Overall, it seems that, when all the investable assets we consider are available for dynamic trading, the opportunity cost of money illusion is substantial. Our estimate is totally consistent with the findings of Stephens and Tyran (2016), who estimated that 10-year portfolio returns were about

10 percentage points lower for Danish money-illusioned individuals.

Perceived utility loss due to the unavailability of inflation-indexed bonds for different degrees of money illusion Figure 7a shows the optimal portfolio strategy when the investor has no access to inflation-indexed bonds. The effect of an increasing degree of money illusion on the portfolio weights is comparable to that already noted for the conservative investor, with little effect on the nominal bond position and a reduction in the stock position.

[Figure 7 about here.]

The most interesting result is shown, however, in the bottom-right panel of Figure 7a, where the annualized certainty equivalent loss due to the exclusion of the inflation-indexed bond from the investable universe is shown. The opportunity cost of not having access to the real bond is increasing in the investment horizon and is substantial for a non-illusioned investor ($\alpha = 0$), being equal to 1.25% per annum for a 30-year horizon. However, the opportunity cost of removing the inflation-indexed bond is perceived as negligible by a totally money-illusioned investor ($\alpha = 1$).

It is crucial that, for an illusioned agent, the utility cost of not having access to inflation-indexed bonds is negligible. This may represent an explanation for the low market demand for inflation-protected securities. Although the optimal portfolio weight associated with the inflation-indexed bond may not be exactly zero, because of the fact that an illusioned investor perceives a negligible loss for not investing in inflation-protected securities, her demand for this kind of securities is likely to be low, as they can be effectively substituted by more traditional assets, such as nominal bonds and stocks.

Figure 7b refers to the opposite situation, showing the optimal portfolio strategy and certainty equivalent loss of an investor deprived of the access to nominal bonds. This situation is highly hypothetical, given the enormous size of the nominal bond market, but it is useful to completely grasp the intuition obtained from the previous analysis. The results are complementary to those obtained in the panel above: there is very little effect in terms of portfolio allocation in the inflation-indexed bond, while a money-illusioned investor tends to allocate a smaller fraction of wealth into the stock market. In terms of the opportunity cost of removing nominal bonds from the investable universe, a money-illusioned investor is severely hurt, while a non-illusioned investor, who tends to favor the allocation into inflation-indexed bonds, is significantly less affected.

Synthesis It seems that money illusion significantly affects the optimal portfolio strategy of long-term investors, and in particular the positions taken in nominal and inflation-indexed bonds. The utility cost of money illusion, if evaluated from the point of view of a fully rational and non-illusioned investor, is significant. Although the optimal allocation of a money-illusioned investor also comprises an investment in inflation-indexed bonds, we show that by excluding inflation-indexed bonds from the investable universe, the utility loss perceived by an illusioned investor is very small, as her perceived expected utility substituting real bonds with nominal bonds is almost unchanged. In Appendix C, we prove that these results are robust to variations of the realized inflation risk premium, which is a quantity which previous studies have shown to be difficult to estimate.

5 Additional empirical findings: time-varying risk premia

In this section, we present additional empirical results, which support the evidence obtained in the previous section for the setting where risk premia are constant, by considering a setting where risk premia are allowed to be time-varying. We focus on the case of the moderate investor ($\gamma = 10$), considering the specification where risk premia are time-varying and their volatilities have been constrained, as specified in Section 3.3. The parameter estimates used are those in Table 1b. We relegate to Appendix D the analysis based on the specification with time-varying risk premia without any volatility constraint. The analysis confirms the conclusions we draw in this section, by providing qualitatively compatible results, but it unrealistically overstates the optimal dynamic portfolio positions and the certainty equivalent returns, facts for which we provide an explanation.

Portfolio strategy Figure 8 shows the optimal portfolio strategy when investors have access to the full investment universe. For short investment horizons, the optimal strategy is virtually identical to the case where risk premia are constant (Figure 6). When the horizon is increased beyond 5 years, the impact of time-varying risk premia seems to shift the optimal portfolio from nominal bonds to inflation-indexed bonds. For a 30-year horizon, the increase of the weight in the real bond is between 30% and 50%, corresponding to an approximately equivalent reduction of the weight in the nominal bond. This is true for any value of α, which suggests that there is little interaction between money illusion and the intertemporal hedging demands arising from capturing the time variation in risk premia.

[Figure 8 about here.]

Utility loss due to money illusion The welfare analysis in the graph at the bottom of Figure 8, showing the certainty equivalent loss attributable to money illusion, confirms that the welfare effect of money illusion is substantial. The loss is very similar in pattern and size to the case of constant risk premia. In fact, the annualized loss is about 1% for a totally illusioned investor with a 10-year horizon w.r.t. a rational investor.[16]

Perceived utility loss due to the unavailability of inflation-indexed bonds for different degrees of money illusion We consider now the case where one of the bonds is removed from the investable universe. Figure 9a shows the case where the inflation-indexed bond is unavailable. In terms of portfolio positions, qualitatively, the effect of money illusion on the optimal allocation is similar to the case with constant risk premia (Figure 7a), although in this case, the effect of

[16]Our analysis is focused on relative losses, not on the absolute value of certainty equivalent returns, which is *ex-ante* higher when accommodating time-varying risk premia. We verify that, for investment horizons beyond 5 years, our specification of time-varying premia entails a certainty equivalent gain of around 0.2% per annum w.r.t. the case of constant risk premia.

money illusion on the intertemporal hedging component associated with the nominal bond is more pronounced, leading to a position for the illusioned investor which is higher by 30% to 40% for a horizon longer than 10 years w.r.t. the non-illusioned investor. This difference corresponds to a different degree of leverage, as reflected by the cash position.

[Figure 9 about here.]

The evaluation of the opportunity cost, perceived by investors with different levels of α, of excluding the real bond, is crucial to confirm that a money-illusioned investor does not perceive as useful the availability of inflation-protected securities. Indeed, in this situation, other than for inflation hedging, an investor who times the market also takes positions in the assets to also hedge future variations in the risk premia. As can be noticed in Figure 9a, the opportunity cost is substantial for the fully rational investor ($\alpha = 0$). The annualized loss increases steadily with the investment horizon, being equal to 0.5% per annum for a 10-year horizon, 1.1% for a 20-year horizon and 1.7% for a 30-year horizon. However, considering a partially illusioned investor ($\alpha = 0.5$), the perceived loss is already significantly reduced, reaching a maximum of 1% for a 30-year horizon. A totally illusioned investor ($\alpha = 1$) perceives a significantly lower loss, which is roughly flat and equal to 0.25% per annum up to a 20-year horizon, and increasing to a maximum of 0.6% for a 30-year horizon.

Finally, for the sake of completeness, we show in Figure 9b the complementary case, where the nominal bond is excluded from the investable universe. The highest utility loss is sustained by the illusioned investor (about 1.5% per annum for a 10- to 30-year horizon), while a non-illusioned investor perceives a loss equal to about 0.4% per annum.

In conclusion, our analysis confirms that a money-illusioned investor, even when attempting to time the market by accounting for time-varying premia, perceives a significantly lower opportunity cost of not having access to inflation-indexed instruments than a non-illusioned investor. Among the money-illusioned investors, even those following sophisticated dynamic strategies seem to be able to attain comparable expected utilities by substituting inflation-indexed bonds with more traditional assets, such as nominal bonds and stocks, and therefore have few incentives to enter the inflation-indexed bond market.

6 Conclusions

Several authors have studied the effects of money illusion on financial markets. Modigliani and Cohn (1979) recognized money illusion as the source of major errors in the evaluation of common stocks during periods of anticipated hyperinflation, concluding that these evaluation mistakes were the main cause of a 50% undervaluation of U.S. stock value at the end of 1977. Their analysis was confirmed by Cohen et al. (2005), who tested the effects of money illusion in a CAPM-based framework, identifying an irrational increase of expected returns across all stocks, irrespective of the riskiness (beta) of the stock, during periods of hyperinflation. More recently, Schmeling and Schrimpf (2011) found that survey-based measures of expected inflation are able to predict aggregate stock returns, attributing this phenomenon to money illusion.

The present work shows that the mechanism at play in the aforementioned empirical investigations is also crucial for the decision-making of a long-term investor. Our results confirm some of the findings of Stephens and Tyran (2016), who documented a tendency for money-illusioned investors to shift portfolios toward nominal assets. In particular, an illusioned investor tends to reduce the allocation in inflation-indexed bonds, which intertemporally hedges expected inflation, in favor of nominal bonds. The effect on the stock allocation is marginal. We estimate the cost of money illusion to be around 1% per annum for a moderately risk-averse individual with an investment horizon of 10 years or longer. Our findings do not suffer from the cognitive challenge of experiments aimed at identifying money illusion, as stated by Petersen et al. (2011). We estimate that a fully rational non-illusioned investor suffers from a utility loss that ranges from about 0.5% to 1.5% per annum for investment horizons of between 10 to 30 years. Conversely, the perceived welfare loss of a money-illusioned investor who has no access to inflation-indexed bond is, depending on the specification of the risk premia considered, negligible or significantly lower. This finding identifies money illusion as potentially being at the origin of the scarce demand for TIPS in the market, as money-illusioned investors understate the utility loss entailed by substituting inflation-indexed bonds with nominal bonds.

From the technical standpoint, we have contributed to the literature of dynamic asset allocation relying on dynamic affine term structure models, showing some other aspects of the empirical issues of essentially affine models that have recently been pointed out by Feldhütter et al. (2012) and Sarno et al. (2016). In particular, we implement a simple, but effective, methodology, to reduce the problem of the in-sample overfitting of the risk premia, which typically leads to unrealistic optimal portfolio positions and largely overstates utility losses entailed by suboptimal strategies. The use of this methodology entails the choice of bounds to the volatility of the risk premia in the economy, which we impose by choosing some values dictated by common sense. We leave for future research a formal analysis of this methodology, which would allow a more accurate fine tuning of the restrictions to be imposed in the estimation phase.

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Table 1: Parameter estimates

The tables show the maximum-likelihood estimates of the model parameters. The values in parentheses are the standard errors of the estimates. The sample period runs from January 1999 until January 2016. Panel (a) shows the parameter estimates obtained for the specification with constant risk premia ($\mathbf{\Lambda}_1 = \mathbf{0}$). Panel (b) shows the parameter estimates obtained for the specification with time-varying risk premia and restrictions on the maximum volatility of bond, realized inflation and stock risk premia. The volatility of the risk premia for the 3- and 10-year nominal bonds, and for the 7-year inflation-indexed bond, are imposed to be lower than the volatility of the short-term nominal rate (0.64%). The volatility of the realized inflation risk premium is not higher than 0.5% per annum and the volatility of the equity premium is not higher than 1% per annum.

(a) Constant risk premia

R_0	$\mathbf{R}_1$	π_0	$\boldsymbol{\pi}_1$	$\mathbf{\Theta}$			σ_ϵ^B	σ_ϵ^I
0.0185	0.3626	0.0210	0.0716	0.0639	−0.5484	1.0786	0.0012	0.0007
(0.0001)	(0.0013)	(0.0022)	(0.0054)	(0.0033)	(0.0149)	(0.0396)	(0.0000)	(0.0000)
	−0.4174		−0.8305	0.0215	0.2526	−0.3158		
	(0.0053)		(0.0301)	(0.0019)	(0.0085)	(0.0213)		
	0.3323		−1.1252	−0.0228	0.0953	0.2829		
	(0.0138)		(0.0771)	(0.0019)	(0.0093)	(0.0254)		

$\mathbf{\Lambda}_0$	$\mathbf{\Lambda}_1$			$\mathbf{\Sigma}_X$			$\boldsymbol{\sigma}_P$	$\boldsymbol{\sigma}_S$
−0.6746	0	0	0	0.0209	0.0066	−0.0043	0.0001	0.0380
(0.0332)				(0.0010)	(0.0009)	(0.0006)	(0.0007)	(0.0107)
0.4464	0	0	0	0	0.0123	0.0014	−0.0009	−0.0325
(0.0517)					(0.0006)	(0.0006)	(0.0007)	(0.0108)
−0.1015	0	0	0	0	0	0.0087	−0.0040	−0.0427
(0.0572)						(0.0004)	(0.0007)	(0.0101)
1.3054	0	0	0	0	0	0	0.0090	−0.0181
(0.2718)							(0.0005)	(0.0100)
0.7694	0	0	0	0	0	0	0	0.1413
(0.2765)								(0.0070)

(b) Time-varying risk premia with volatility restrictions

R_0	$\mathbf{R}_1$	π_0	$\boldsymbol{\pi}_1$	$\mathbf{\Theta}$			σ_ϵ^B	σ_ϵ^I
0.0185	0.3626	0.0208	0.0949	0.0642	−0.2302	1.8305	0.0012	0.0007
(0.0001)	(0.0013)	(0.0022)	(0.0395)	(0.0563)	(0.1384)	(0.2640)	(0.0000)	(0.0000)
	−0.4181		−0.7257	0.0031	0.0620	−0.6949		
	(0.0053)		(0.1119)	(0.0426)	(0.1067)	(0.1989)		
	0.3304		−1.5538	−0.0113	0.1566	0.6364		
	(0.0137)		(0.2107)	(0.0327)	(0.0708)	(0.1220)		

$\mathbf{\Lambda}_0$	$\mathbf{\Lambda}_1$			$\mathbf{\Sigma}_X$			$\boldsymbol{\sigma}_P$	$\boldsymbol{\sigma}_S$
−0.6922	−0.0048	−15.7462	−37.4191	0.0204	0.0069	−0.0044	−0.0001	0.0365
(0.0340)	(2.7657)	(6.9130)	(13.1314)	(0.0010)	(0.0009)	(0.0006)	(0.0007)	(0.0105)
0.4902	1.5536	25.2406	53.6283	0	0.0119	0.0015	−0.0008	−0.0302
(0.0547)	(3.9027)	(12.1339)	(23.5016)		(0.0006)	(0.0006)	(0.0007)	(0.0106)
−0.1367	−1.6353	−20.0336	−71.5528	0	0	0.0085	−0.0040	−0.0431
(0.0595)	(3.8535)	(10.7760)	(20.6652)			(0.0004)	(0.0006)	(0.0099)
1.2851	2.1164	4.5350	−79.6940	0	0	0	0.0088	−0.0190
(0.2752)	(4.2971)	(14.0612)	(28.0800)				(0.0004)	(0.0099)
0.7610	−4.2405	5.6160	−16.8258	0	0	0	0	0.1401
(0.2772)	(3.2702)	(7.0408)	(14.0275)					(0.0070)

Table 2: Historical and model-implied summary statistics.

The tables show annualized historical and model-implied means and volatilities of bond yields, equity logarithmic returns and realized inflation. The tables also show the model-implied means and volatilities of bond risk premia and equity risk premium, as well as of the nominal risk-free rate and the expected inflation.

(a) Constant risk premia

Time series	Mean value		Volatility	
	Estimation	Data	Estimation	Data
3M nominal yield	1.91%	1.87%	0.63%	0.64%
6M nominal yield	1.97%	1.97%	0.62%	0.61%
1Y nominal yield	2.10%	2.15%	0.63%	0.74%
2Y nominal yield	2.37%	2.38%	0.71%	0.85%
3Y nominal yield	2.64%	2.63%	0.80%	0.91%
5Y nominal yield	3.14%	3.12%	0.93%	0.96%
7Y nominal yield	3.55%	3.54%	0.96%	0.96%
10Y nominal yield	3.99%	4.01%	0.91%	0.95%
5Y real yield	1.29%	1.28%	0.92%	0.97%
7Y real yield	1.51%	1.54%	0.88%	0.86%
10Y real yield	1.81%	1.80%	0.84%	0.77%
Log realized inflation	2.09%	2.17%	0.99%	1.07%
Equity log returns	5.55%	4.98%	15.69%	15.72%
3M nominal risk premium	0.12%		0%	
6M nominal risk premium	0.25%		0%	
1Y nominal risk premium	0.51%		0%	
2Y nominal risk premium	1.07%		0%	
3Y nominal risk premium	1.62%		0%	
5Y nominal risk premium	2.55%		0%	
7Y nominal risk premium	3.20%		0%	
10Y nominal risk premium	3.71%		0%	
5Y real risk premium	2.22%		0%	
7Y real risk premium	2.74%		0%	
10Y real risk premium	3.28%		0%	
Realized inflation risk premium	1.18%		0%	
Equity risk premium	4.93%		0%	
Nominal risk-free rate	1.85%		0.65%	
Expected inflation	2.10%		1.54%	

(b) Time-varying risk premia with volatility restrictions

Time series	Mean value		Volatility	
	Estimation	Data	Estimation	Data
3M nominal yield	1.91%	1.87%	0.59%	0.64%
6M nominal yield	1.97%	1.97%	0.59%	0.61%
1Y nominal yield	2.10%	2.15%	0.60%	0.74%
2Y nominal yield	2.37%	2.38%	0.69%	0.85%
3Y nominal yield	2.64%	2.63%	0.79%	0.91%
5Y nominal yield	3.14%	3.12%	0.92%	0.96%
7Y nominal yield	3.55%	3.54%	0.95%	0.96%
10Y nominal yield	3.99%	4.01%	0.90%	0.95%
5Y real yield	1.29%	1.28%	0.90%	0.97%
7Y real yield	1.51%	1.54%	0.87%	0.86%
10Y real yield	1.81%	1.80%	0.83%	0.77%
Log realized inflation	2.08%	2.17%	0.97%	1.07%
Equity log returns	5.46%	4.98%	15.52%	15.72%
3M nominal risk premium	0.12%		0.14%	
6M nominal risk premium	0.25%		0.25%	
1Y nominal risk premium	0.51%		0.44%	
2Y nominal risk premium	1.07%		0.63%	
3Y nominal risk premium	1.62%		0.64%	
5Y nominal risk premium	2.55%		0.38%	
7Y nominal risk premium	3.20%		0.21%	
10Y nominal risk premium	3.70%		0.64%	
5Y real risk premium	2.20%		0.76%	
7Y real risk premium	2.71%		0.64%	
10Y real risk premium	3.26%		0.34%	
Realized inflation risk premium	1.15%		0.50%	
Equity risk premium	4.81%		1.00%	
Nominal risk-free rate	1.85%		0.61%	
Expected inflation	2.08%		1.76%	

Table 3: Correlations between asset returns and economic variables.

Panel (a) shows the unconditional correlations between nominal and real bond returns, stock returns and realized inflation, calculated from the monthly time series. Panels (b) and (c) report the one-month conditional pairwise correlations between nominal and real bond returns, stock returns, realized inflation, nominal interest rate, expected inflation and real interest rate.

(a) Data

	3M nom	1Y nom	2Y nom	5Y nom	10Y nom	5Y real	10Y real	Equity	CPI
3M nom	1.000								
1Y nom	0.658	1.000							
2Y nom	0.471	0.924	1.000						
5Y nom	0.257	0.708	0.889	1.000					
10Y nom	0.108	0.499	0.669	0.904	1.000				
5Y real	−0.000	0.311	0.409	0.501	0.476	1.000			
10Y real	−0.009	0.322	0.440	0.623	0.682	0.910	1.000		
Equity	−0.160	−0.299	−0.352	−0.323	−0.253	0.038	0.000	1.000	
CPI	−0.124	−0.150	−0.139	−0.190	−0.234	0.350	0.156	0.070	1.000

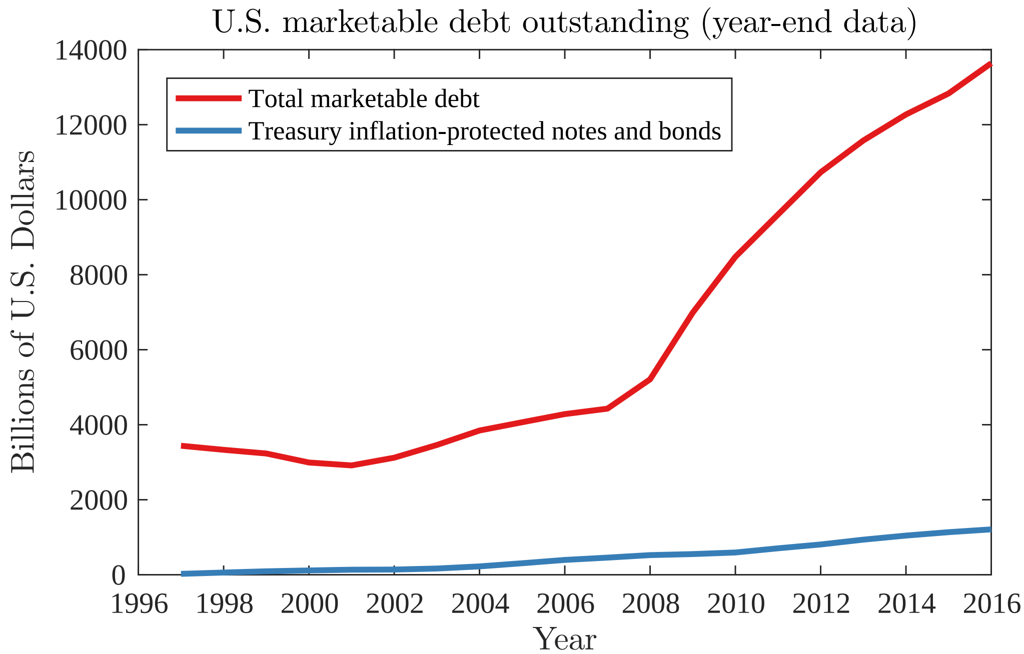
(b) Constant risk premia

	3M nom	1Y nom	2Y nom	5Y nom	10Y nom	5Y real	10Y real	Equity	CPI	R	π	r
3M nom	1.000											
1Y nom	0.926	1.000										
2Y nom	0.744	0.939	1.000									
5Y nom	0.398	0.705	0.904	1.000								
10Y nom	0.209	0.537	0.782	0.969	1.000							
5Y real	0.166	0.274	0.366	0.494	0.618	1.000						
10Y real	0.114	0.319	0.492	0.686	0.807	0.958	1.000					
Equity	−0.203	−0.295	−0.335	−0.306	−0.237	0.089	−0.002	1.000				
CPI	0.050	−0.048	−0.122	−0.158	−0.115	0.270	0.156	0.046	1.000			
R	−0.978	−0.860	−0.634	−0.254	−0.064	−0.120	−0.028	0.156	−0.091	1.000		
π	−0.340	−0.404	−0.385	−0.216	−0.013	0.715	0.561	0.346	0.380	0.296	1.000	
r	−0.073	0.046	0.124	0.114	−0.015	−0.795	−0.595	−0.292	−0.434	0.128	−0.910	1.000

(c) Time-varying risk premia with volatility restrictions

	3M nom	1Y nom	2Y nom	5Y nom	10Y nom	5Y real	10Y real	Equity	CPI	R	π	r
3M nom	1.000											
1Y nom	0.924	1.000										
2Y nom	0.741	0.939	1.000									
5Y nom	0.399	0.710	0.907	1.000								
10Y nom	0.209	0.544	0.787	0.970	1.000							
5Y real	0.127	0.255	0.362	0.507	0.633	1.000						
10Y real	0.087	0.308	0.491	0.693	0.813	0.961	1.000					
Equity	−0.195	−0.287	−0.328	−0.302	−0.238	0.084	−0.006	1.000				
CPI	0.050	−0.052	−0.128	−0.165	−0.123	0.270	0.153	0.046	1.000			
R	−0.974	−0.844	−0.613	−0.238	−0.052	−0.100	−0.012	0.133	−0.113	1.000		
π	−0.341	−0.480	−0.523	−0.417	−0.237	0.567	0.365	0.381	0.423	0.224	1.000	
r	−0.022	0.080	0.139	0.105	−0.029	−0.792	−0.598	−0.293	−0.449	0.113	−0.911	1.000

Figure 1: U.S. marketable debt outstanding.



Source: Bureau of the Fiscal Service – Treasury Bulletin

Figure 2: Time series of macroeconomic variables, risk premia, mean-variance portfolio positions and the maximum achievable Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond, the stock index and $\gamma = 10$).

(a) Constant risk premia

(b) Time-varying risk premia with volatility restrictions

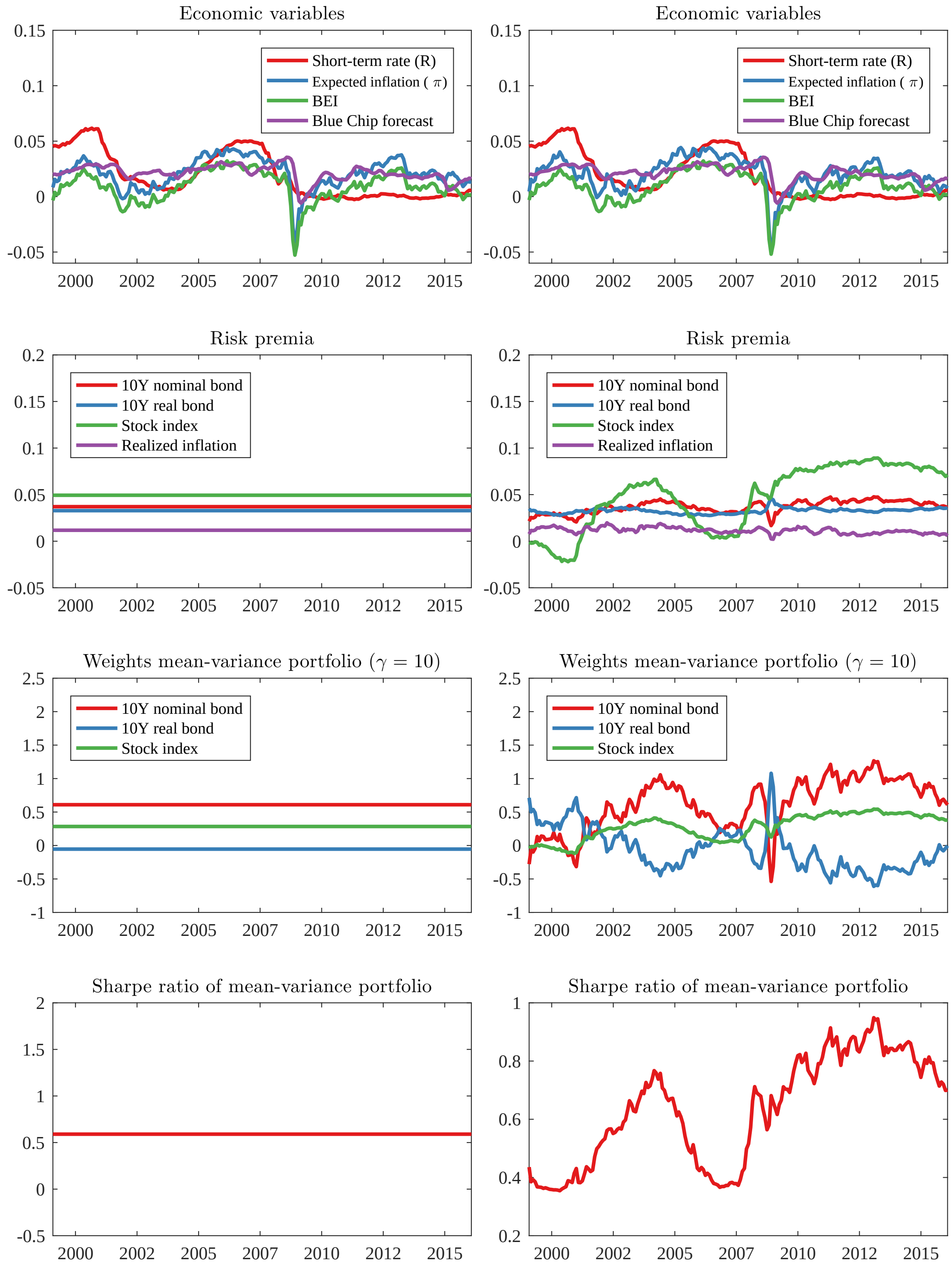


Figure 3: Optimal portfolio strategy for $\gamma \to \infty$. Constant risk premia.

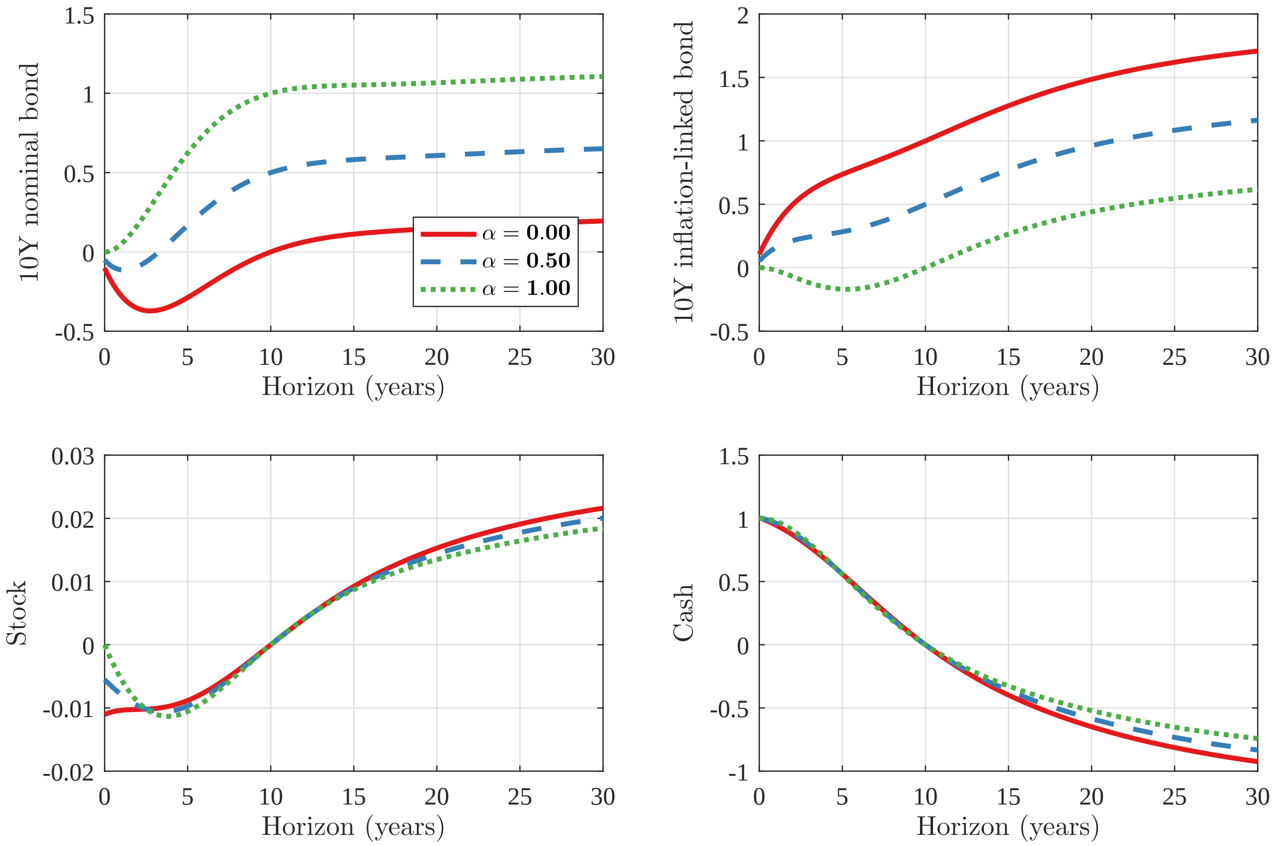


Figure 4: Intertemporal hedging components projected onto (and orthogonal to) expected inflation and the nominal short-term interest rate for $\gamma \to \infty$. Constant risk premia.

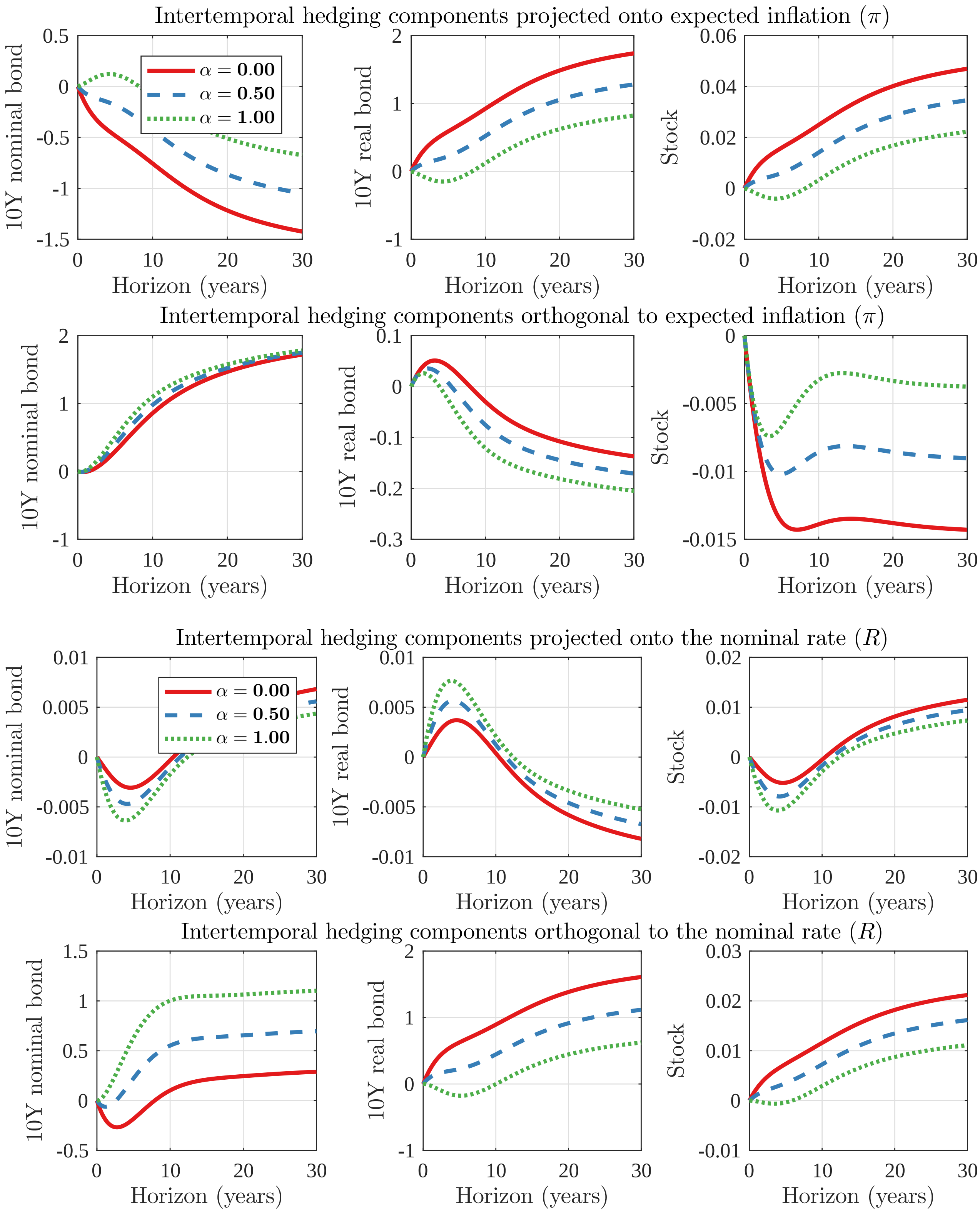
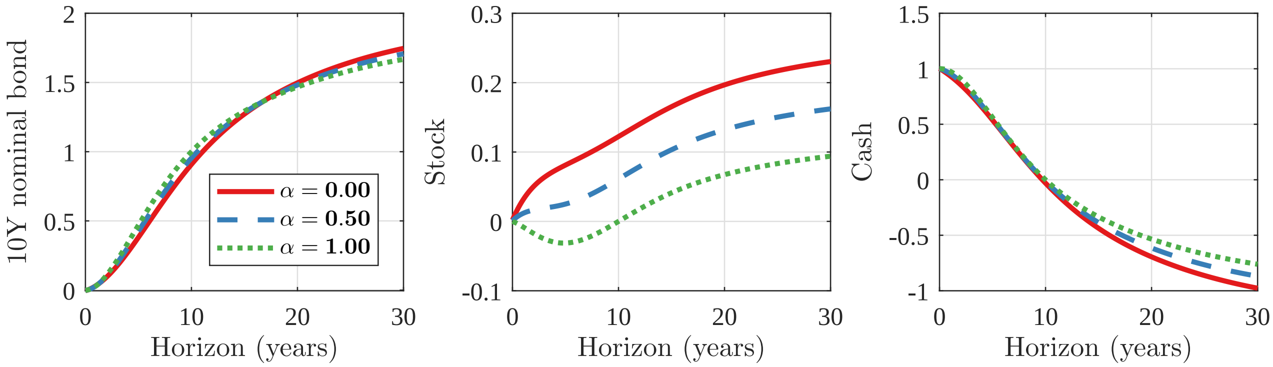


Figure 5: Optimal portfolio strategy for $\gamma \to \infty$ when the 10-year inflation-indexed or nominal bond is excluded from the investable universe. Constant risk premia.

(a) Inflation-indexed bond removed from investable universe



(b) Nominal bond removed from investable universe

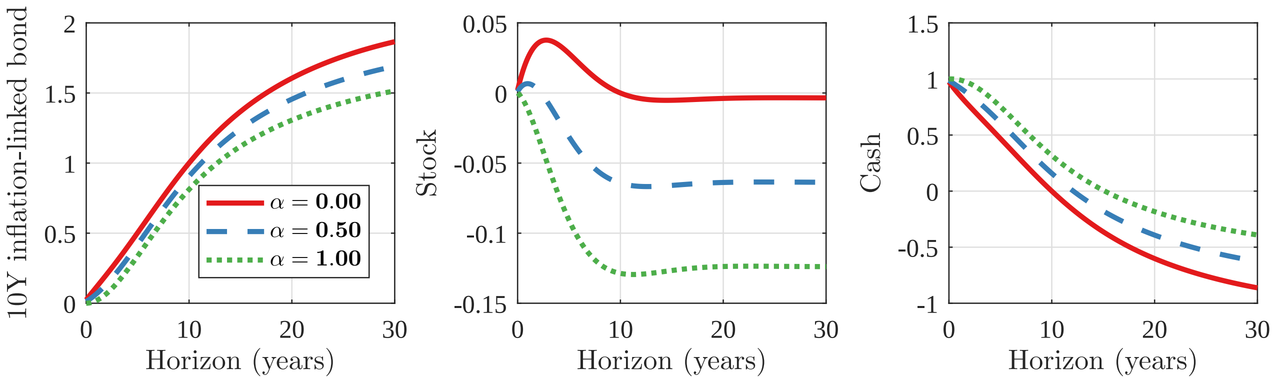


Figure 6: Optimal portfolio strategy for $\gamma = 10$. Utility loss with respect to a non-illusioned investor. Constant risk premia.

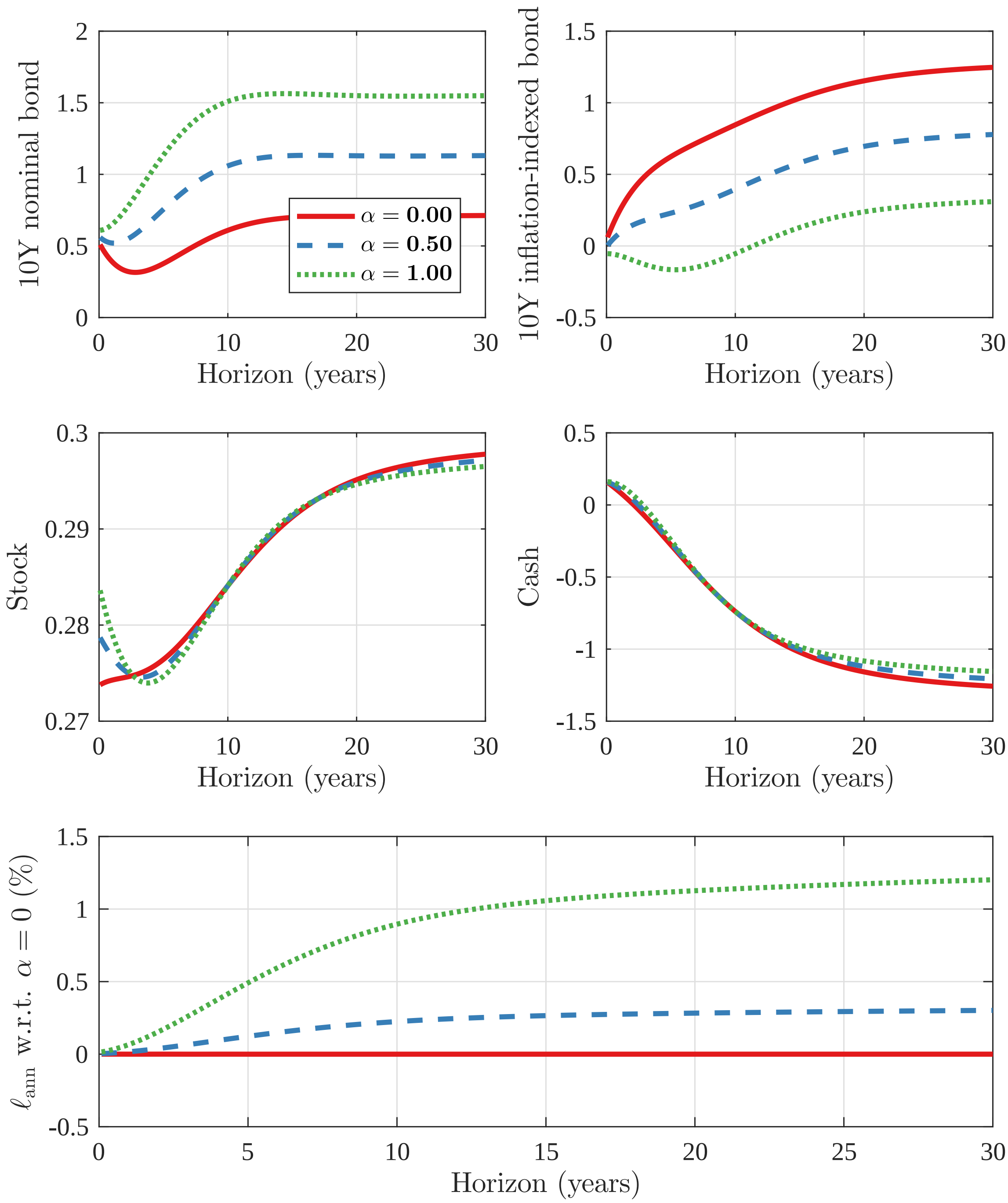
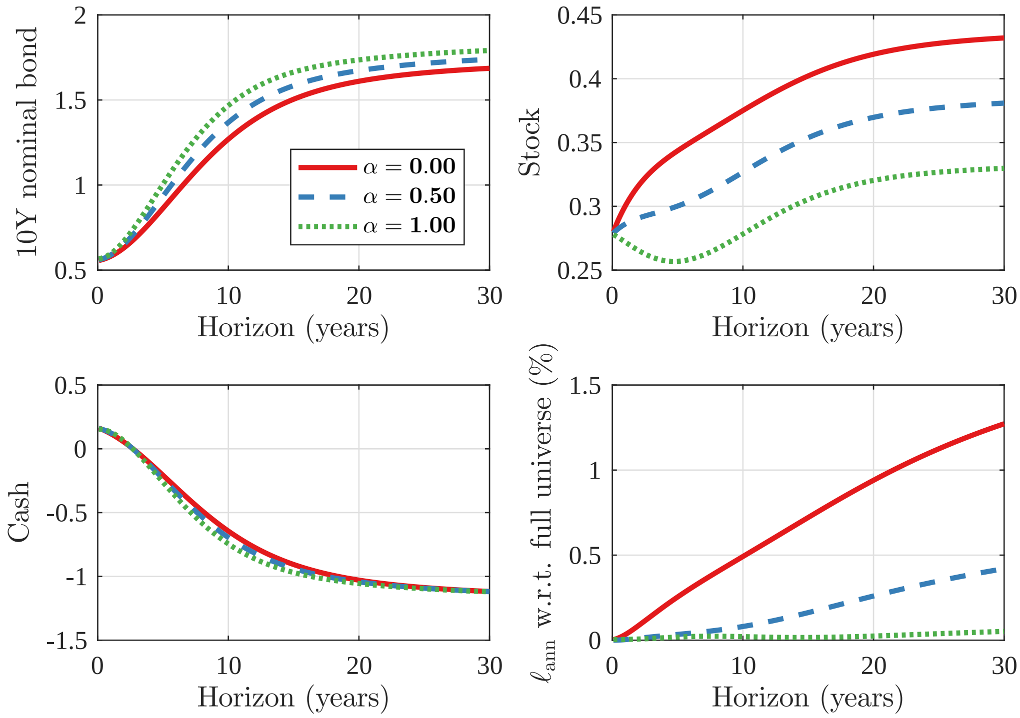


Figure 7: Optimal portfolio strategy for $\gamma = 10$ when the 10-year inflation-indexed or nominal bond is excluded from the investable universe. Utility loss with respect to the case where the full investable universe is available. Constant risk premia.

(a) Inflation-indexed bond removed from investable universe



(b) Nominal bond removed from investable universe

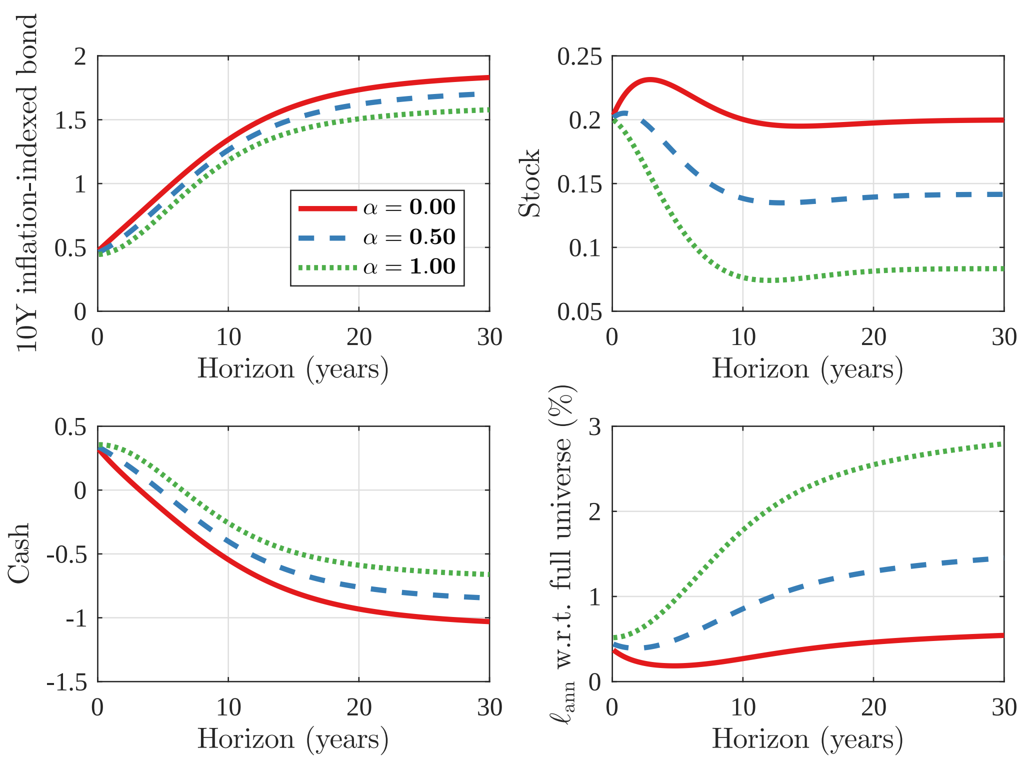


Figure 8: Optimal portfolio strategy for $\gamma = 10$. Utility loss with respect to a non-illusioned investor. Time-varying risk premia (with volatility constraints).

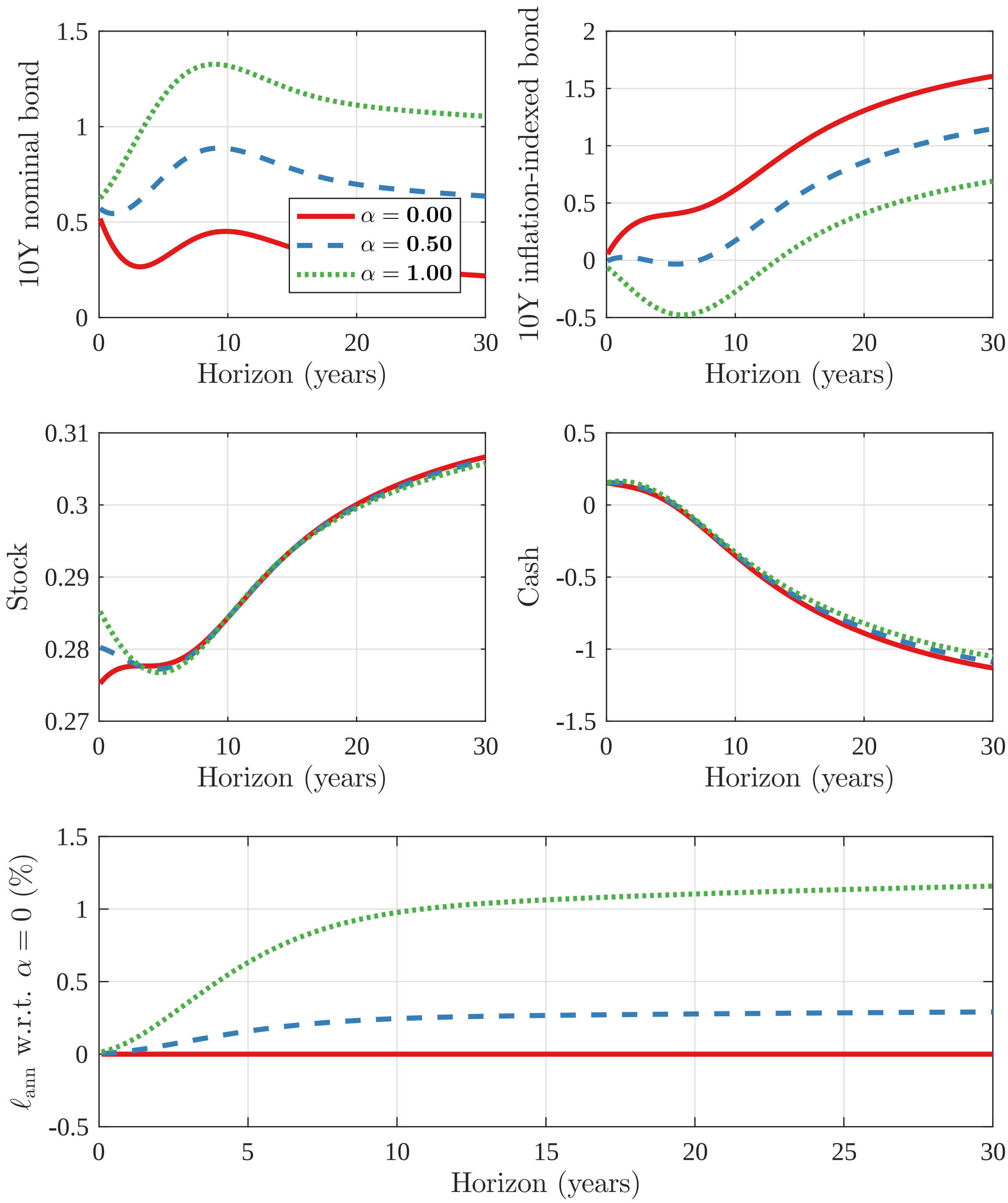
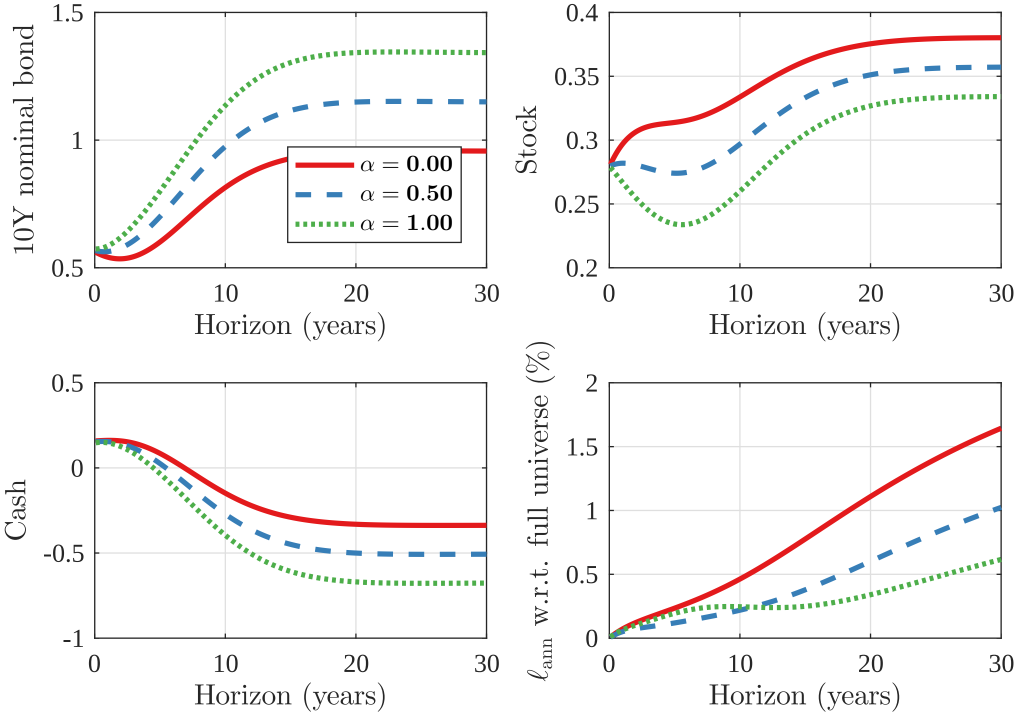
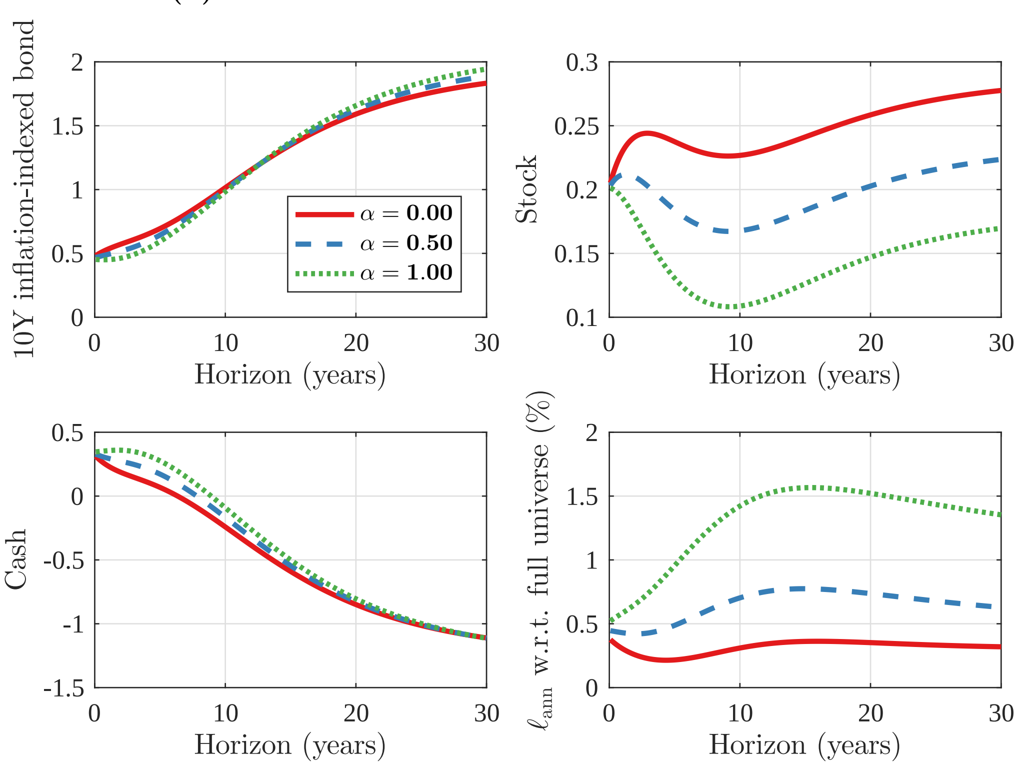


Figure 9: Optimal portfolio strategy for $\gamma = 10$ when the 10-year inflation-indexed or nominal bond is excluded from the investable universe. Utility loss with respect to the case where the full investable universe is available. Time-varying risk premia (with volatility constraints).

(a) Inflation-indexed bond removed from investable universe



(b) Nominal bond removed from investable universe



Appendix to

Money Illusion:

A Rationale for the TIPS Puzzle

March 2018

This Appendix is organized as follows. In Section A, we derive the pricing equations for nominal bonds (A.1) and real bonds (A.2). The derivation of the optimal portfolio strategy is in Section B. In particular, we derive the optimal portfolio strategy for complete and incomplete markets (B.1), the optimal portfolio strategy for an infinitely risk-averse individual and constant market prices of risk (B.2), the utility from suboptimal strategies (B.3) and the certainty equivalent utility loss (B.4). In Section C, we discuss the empirical findings obtained imposing a zero unexpected inflation risk premium. Finally, in Section D, we elaborate on the specification with time-varying risk premia where no volatility constraints are imposed. We discuss the parameter estimates (D.1) and the portfolio strategy and welfare (D.2).

A Bond pricing

A.1 Nominal bond pricing

We conjecture that the nominal price of a zero-coupon nominal bond has the following functional form:

$$B\left(\mathbf{X}_t,\tau\right) = e^{-y^n(\mathbf{X}_t,\tau)\tau} = e^{A_0^N(\tau)+\mathbf{A}_1^N(\tau)\mathbf{X}_t}.$$

From no-arbitrage arguments, the nominal price B of a nominal bond satisfies the following PDE:

$$B_{\mathbf{X}}\boldsymbol{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right)+\frac{1}{2}\mathrm{tr}\left(B_{\mathbf{XX}}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right)+B_t-R_tB = B_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{X}}'\Lambda_t$$

Computing the derivatives:

$$\begin{aligned} B_{\mathbf{X}} &= \mathbf{A}_1^N B \\ B_{\mathbf{XX}} &= \left(\mathbf{A}_1^N\right)'\mathbf{A}_1^N B \end{aligned}$$

$$B_t \quad = \quad -B_\tau = -\left(\frac{\partial}{\partial\tau}A_0^N(\tau) + \frac{\partial}{\partial\tau}\mathbf{A}_1^N(\tau)\,\mathbf{X}_t\right)B,$$

substituting into the PDE the derivatives just computed, as well as the expressions for R_t and $\mathbf{\Lambda}_t$:

$$\begin{aligned}
\mathbf{A}_1^N\mathbf{\Sigma}_\mathbf{X}'(\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1\mathbf{X}_t) \quad &= \quad \mathbf{A}_1^N\mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) + \frac{1}{2}\mathbf{A}_1^N\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Sigma}_\mathbf{X}\left(\mathbf{A}_1^N\right)' - \left(\frac{\partial}{\partial\tau}A_0^N(\tau) + \frac{\partial}{\partial\tau}\mathbf{A}_1^N(\tau)\,\mathbf{X}_t\right) \\
&\quad - \left(R_0 + \mathbf{R}_1'\mathbf{X}_t\right).
\end{aligned}$$

Matching the homogeneous term and the term in $\mathbf{X}_t$:

$$\frac{\partial}{\partial\tau}A_0^N(\tau) \quad = \quad \mathbf{A}_1^N\left(\mathbf{\Theta}\bar{\mathbf{X}} - \mathbf{\Sigma}_\mathbf{X}'\mathbf{\Lambda}_0\right) + \frac{1}{2}\mathbf{A}_1^N\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Sigma}_\mathbf{X}\left(\mathbf{A}_1^N\right)' - R_0, \tag{A.1}$$

$$\frac{\partial}{\partial\tau}\mathbf{A}_1^N(\tau) \quad = \quad -\mathbf{A}_1^N\left(\mathbf{\Theta} + \mathbf{\Sigma}_\mathbf{X}'\mathbf{\Lambda}_1\right) - \mathbf{R}_1'. \tag{A.2}$$

The nominal rate is given considering the limit for $\tau \to 0$:

$$R_t = R_0 + \mathbf{R}_1'\mathbf{X}_t.$$

The dynamics of the price of a zero-coupon nominal bond expiring at time T is:

$$\begin{aligned}
\frac{\mathrm{d}B(\mathbf{X}_t,\tau)}{B(\mathbf{X}_t,\tau)} \quad &= \quad \left[-\left(\frac{\partial}{\partial\tau}A_0^N(\tau) + \frac{\partial}{\partial\tau}\mathbf{A}_1^N(\tau)\,\mathbf{X}_t\right) + \mathbf{A}_1^N\mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) + \frac{1}{2}\mathbf{A}_1^N\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Sigma}_\mathbf{X}\left(\mathbf{A}_1^N\right)'\right]\mathrm{d}t \\
&\quad + \mathbf{A}_1^N\mathbf{\Sigma}_\mathbf{X}'\mathrm{d}\mathbf{z}_t.
\end{aligned}$$

Substituting for the expressions (A.1) and (A.2):

$$\begin{aligned}
\frac{\mathrm{d}B(\mathbf{X}_t,P_t,\tau)}{B(\mathbf{X}_t,P_t,\tau)} \quad &= \quad -\left[\begin{array}{c} \mathbf{A}_1^N\left(\mathbf{\Theta}\bar{\mathbf{X}} - \mathbf{\Sigma}_\mathbf{X}'\mathbf{\Lambda}_0\right) + \frac{1}{2}\mathbf{A}_1^N\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Sigma}_\mathbf{X}\left(\mathbf{A}_1^N\right)' - R_0 \\ -\mathbf{A}_1^N\left(\mathbf{\Theta} + \mathbf{\Sigma}_\mathbf{X}'\mathbf{\Lambda}_1\right)\mathbf{X}_t - \mathbf{R}_1'\mathbf{X}_t \end{array}\right]\mathrm{d}t \\
&\quad + \left[\mathbf{A}_1^N\mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) + \frac{1}{2}\mathbf{A}_1^N\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Sigma}_\mathbf{X}\left(\mathbf{A}_1^N\right)'\right]\mathrm{d}t \\
&\quad + \mathbf{A}_1^N\mathbf{\Sigma}_\mathbf{X}'\mathrm{d}\mathbf{z}_t.
\end{aligned}$$

We finally obtain:

$$\frac{\mathrm{d}B(\mathbf{X}_t,P_t,\tau)}{B(\mathbf{X}_t,P_t,\tau)} \quad = \quad \left[R_0 + \mathbf{R}_1'\mathbf{X}_t + \mathbf{A}_1^N\mathbf{\Sigma}_\mathbf{X}'(\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1\mathbf{X}_t)\right]\mathrm{d}t + \mathbf{A}_1^N\mathbf{\Sigma}_\mathbf{X}'\mathrm{d}\mathbf{z}_t.$$

The instantaneous time-varying risk premium is given by:

$$\mathbf{A}_1^N \mathbf{\Sigma}_{\mathbf{X}}' \left(\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 \mathbf{X}_t\right).$$

A.2 Real bond pricing

We conjecture that the nominal price of a zero-coupon inflation-indexed bond has the following functional form:

$$I\left(\mathbf{X}_t, P_t, \tau\right) = P_t e^{-y^r(\mathbf{X}_t,\tau)\tau} = P_t e^{A_0^I(\tau) + \mathbf{A}_1^I(\tau)\mathbf{X}_t}.$$

From no-arbitrage arguments, the nominal price I of an inflation-indexed bond satisfies the following PDE:

$$\begin{aligned}\left(I_{\mathbf{X}} \mathbf{\Sigma}_{\mathbf{X}}' + I_P \boldsymbol{\sigma}_P' P_t\right) \Lambda_t &= I_{\mathbf{X}} \mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) + I_P P_t \pi_t + \frac{1}{2}\operatorname{tr}\left(I_{\mathbf{XX}} \mathbf{\Sigma}_{\mathbf{X}}' \mathbf{\Sigma}_{\mathbf{X}}\right) + I_{\mathbf{X}P} \mathbf{\Sigma}_{\mathbf{X}}' \boldsymbol{\sigma}_P P_t \\ &\quad + \frac{1}{2} I_{PP} \boldsymbol{\sigma}_P' \boldsymbol{\sigma}_P P_t^2 + I_t - R_t I.\end{aligned}$$

Computing the derivatives:

$$\begin{aligned}I_{\mathbf{X}} &= \mathbf{A}_1^I I, \\ I_P &= \frac{I}{P_t}, \\ I_{\mathbf{XX}} &= \left(\mathbf{A}_1^I\right)' \mathbf{A}_1^I I, \\ I_{\mathbf{X}P} &= \mathbf{A}_1^I \frac{I}{P_t}, \\ I_{PP} &= 0, \\ I_t &= -I_\tau = -\left(\frac{\partial}{\partial\tau} A_0^I(\tau) + \frac{\partial}{\partial\tau}\mathbf{A}_1^I(\tau)\mathbf{X}_t\right) I.\end{aligned}$$

Substituting into the PDE the derivatives just computed, as well as the expressions for R_t, π_t and $\mathbf{\Lambda}_t$, yields:

$$\begin{aligned}\left(\mathbf{A}_1^I \mathbf{\Sigma}_{\mathbf{X}}' + \boldsymbol{\sigma}_P'\right)\left(\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 \mathbf{X}_t\right) &= \mathbf{A}_1^I \mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) + \left(\pi_0 + \boldsymbol{\pi}_1' \mathbf{X}_t\right) + \frac{1}{2}\mathbf{A}_1^I \mathbf{\Sigma}_{\mathbf{X}}' \mathbf{\Sigma}_{\mathbf{X}} \left(\mathbf{A}_1^I\right)' \\ &\quad + \mathbf{A}_1^I \mathbf{\Sigma}_{\mathbf{X}}' \boldsymbol{\sigma}_P - \left(\frac{\partial}{\partial\tau} A_0^I(\tau) + \frac{\partial}{\partial\tau}\mathbf{A}_1^I(\tau)\mathbf{X}_t\right) \\ &\quad - \left(R_0 + \mathbf{R}_1' \mathbf{X}_t\right).\end{aligned}$$

Matching the homogeneous term and the term in $\mathbf{X}_t$:

$$\begin{aligned}
\frac{\partial}{\partial\tau}A_0^I(\tau) &= \mathbf{A}_1^I\left(\boldsymbol{\Theta}\bar{\mathbf{X}} - \boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Lambda}_0\right) + \mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P + \frac{1}{2}\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^I\right)', -R_0 + \pi_0 - \boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_0, && \text{(A.3)}\\
\frac{\partial}{\partial\tau}\mathbf{A}_1^I(\tau) &= -\mathbf{A}_1^I\left(\boldsymbol{\Theta} + \boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Lambda}_1\right) - \mathbf{R}_1' + \boldsymbol{\pi}_1' - \boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_1. && \text{(A.4)}
\end{aligned}$$

The real rate is given considering the limit for $\tau \to 0$:

$$r_t = R_0 - \pi_0 + \boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_0 + \left(\mathbf{R}_1' - \boldsymbol{\pi}_1' + \boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_1\right)\mathbf{X}_t = R_t - \pi_t + \boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_t.$$

The drift of the price of a zero-coupon real bond expiring at time T can be computed as:

$$\begin{aligned}
\frac{\mathrm{d}I(\mathbf{X}_t, P_t, \tau)}{I(\mathbf{X}_t, P_t, \tau)} &= \left[\begin{array}{c} -\left(\frac{\partial}{\partial\tau}A_0^I(\tau) + \frac{\partial}{\partial\tau}\mathbf{A}_1^I(\tau)\mathbf{X}_t\right) + \mathbf{A}_1^I\boldsymbol{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) \\ + \left(\pi_0 + \boldsymbol{\pi}_1'\mathbf{X}_t\right) + \frac{1}{2}\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^I\right)' + \mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P \end{array}\right]\mathrm{d}t \\
&\quad + \mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\mathrm{d}\mathbf{z}_t + \boldsymbol{\sigma}_P'\mathrm{d}\mathbf{z}_t.
\end{aligned}$$

Substituting for the expressions (A.3) and (A.4):

$$\begin{aligned}
\frac{\mathrm{d}I(\mathbf{X}_t, P_t, \tau)}{I(\mathbf{X}_t, P_t, \tau)} &= -\left[\begin{array}{c} \mathbf{A}_1^I\left(\boldsymbol{\Theta}\bar{\mathbf{X}} - \boldsymbol{\Sigma}_{\mathbf{X}}'\left(\boldsymbol{\Lambda}_0 - \boldsymbol{\sigma}_P\right)\right) + \frac{1}{2}\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^I\right)' \\ -R_0 + \pi_0 - \boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_0 - \mathbf{A}_1^I\left(\boldsymbol{\Theta} + \boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Lambda}_1\right)\mathbf{X}_t \\ -\mathbf{R}_1'\mathbf{X}_t + \boldsymbol{\pi}_1'\mathbf{X}_t - \boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_1\mathbf{X}_t \end{array}\right]\mathrm{d}t \\
&\quad + \left[\begin{array}{c} \mathbf{A}_1^I\boldsymbol{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) + \left(\pi_0 + \boldsymbol{\pi}_1'\mathbf{X}_t\right) + \frac{1}{2}\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^I\right)' \\ + \mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P \end{array}\right]\mathrm{d}t \\
&\quad + \mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\mathrm{d}\mathbf{z}_t + \boldsymbol{\sigma}_P'\mathrm{d}\mathbf{z}_t.
\end{aligned}$$

We finally obtain:

$$\begin{aligned}
\frac{\mathrm{d}I(\mathbf{X}_t, P_t, \tau)}{I(\mathbf{X}_t, P_t, \tau)} &= \left[R_0 + \mathbf{R}_1'\mathbf{X}_t + \left(\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}' + \boldsymbol{\sigma}_P'\right)\left(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1\mathbf{X}_t\right)\right]\mathrm{d}t \\
&\quad + \left(\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}' + \boldsymbol{\sigma}_P'\right)\mathrm{d}\mathbf{z}_t.
\end{aligned}$$

The instantaneous time-varying risk premium is given by:

$$\left(\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}' + \boldsymbol{\sigma}_P'\right)\left(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1\mathbf{X}_t\right)$$

B Portfolio strategy

B.1 Optimal portfolio strategy in incomplete markets

As is well known, the value function can be expressed as:

$$J\left(W_t, t\right) = \frac{W_t^{1-\gamma} P_t^{-(1-\alpha)(1-\gamma)}}{1-\gamma} f\left(\mathbf{X}_t, t, T\right) \tag{B.5}$$

With the martingale approach, it can be shown that:

$$W_t = l^{-\frac{1}{\gamma}} \Phi_t^{-\frac{1}{\gamma}} P_t^{-(1-\alpha)\frac{1-\gamma}{\gamma}} \mathrm{E}_t \left[\left(\frac{\Phi_T}{\Phi_t} \left(\frac{P_T}{P_t} \right)^{(1-\alpha)} \right)^{1-\frac{1}{\gamma}} \right], \tag{B.6}$$

where $\Phi_0 = 1$, $P_0 = 1$ and:

$$l^{-\frac{1}{\gamma}} = \frac{W_0}{\mathrm{E}_0 \left[\left(\frac{\Phi_T}{\Phi_0} \left(\frac{P_T}{P_0} \right)^{(1-\alpha)} \right)^{1-\frac{1}{\gamma}} \right]}.$$

We rewrite wealth as:

$$W_t = W_0 \Phi_t^{-\frac{1}{\gamma}} P_t^{-(1-\alpha)\frac{1-\gamma}{\gamma}} \frac{F\left(\mathbf{X}_t, t, T\right)}{F\left(\mathbf{X}_0, 0, T\right)} \equiv G\left(\Phi_t, P_t, \mathbf{X}_t, t, T\right), \tag{B.7}$$

where:

$$F\left(\mathbf{X}_t, t, T\right) = \mathrm{E}_t \left[\left(\frac{\Phi_T}{\Phi_t} \left(\frac{P_T}{P_t} \right)^{(1-\alpha)} \right)^{1-\frac{1}{\gamma}} \right].$$

Computing the first-order condition of the equation of the value function (B.5) and solving by W_t, by comparison with (B.6), it immediately follows that $f\left(\mathbf{X}_t, t, T\right)$ is such that the value function can be expressed as follows:

$$J\left(W_t, t\right) = \frac{W_t^{1-\gamma} P_t^{-(1-\alpha)(1-\gamma)}}{1-\gamma} \left[F\left(\mathbf{X}_t, t, T\right)\right]^{\gamma}, \tag{B.8}$$

The function $F\left(\mathbf{X}_t, t, T\right)$ takes the form:

$$F\left(\mathbf{X}_t, t, T\right) = \exp\left\{ \frac{1}{2} \mathbf{X}_t' \mathbf{B}_3(\tau) \mathbf{X}_t + \mathbf{B}_2(\tau) \mathbf{X}_t + B_1(\tau) \right\}.$$

Remembering that $G \equiv W_t$, the following no-arbitrage relation holds:

$$\mathcal{L}G + G_t - R_t G = \left(-G_\Phi \Phi_t \boldsymbol{\Lambda}_t' + G_P P_t \boldsymbol{\sigma}_P' + G_{\mathbf{X}} \boldsymbol{\Sigma}_{\mathbf{X}}'\right) \boldsymbol{\Lambda}_t, \tag{B.9}$$

where:

$$\begin{aligned}
\mathcal{L}G &= G_\Phi \left(-\Phi_t R_t\right) + G_P \left(P_t \pi_t\right) + G_{\mathbf{X}} \boldsymbol{\Theta} \left(\bar{\mathbf{X}} - \mathbf{X}_t\right) \\
&\quad + G_{\Phi \mathbf{X}} \boldsymbol{\Sigma}_{\mathbf{X}}' \left(-\Phi_t \boldsymbol{\Lambda}_t\right) + G_{P\mathbf{X}} \boldsymbol{\Sigma}_{\mathbf{X}}' \left(P_t \boldsymbol{\sigma}_P\right) + G_{\Phi P} \left(P_t \boldsymbol{\sigma}_P\right)' \left(-\Phi_t \boldsymbol{\Lambda}_t\right) \\
&\quad + \frac{1}{2}\left(G_{\Phi\Phi} \left(-\Phi_t \boldsymbol{\Lambda}_t\right)' \left(-\Phi_t \boldsymbol{\Lambda}_t\right) + G_{PP} \left(P_t \boldsymbol{\sigma}_P\right)' \left(P_t \boldsymbol{\sigma}_P\right) + \operatorname{tr}\left(G_{\mathbf{X}\mathbf{X}} \boldsymbol{\Sigma}_{\mathbf{X}}' \boldsymbol{\Sigma}_{\mathbf{X}}\right)\right).
\end{aligned}$$

Substituting for the partial derivatives of G:

$$\begin{aligned}
\mathcal{L}G &= -\frac{1}{\gamma} \Phi_t^{-1} G \left(-\Phi_t R_t\right) - (1-\alpha) \frac{1-\gamma}{\gamma} P_t^{-1} G \left(P_t \pi_t\right) + \frac{F_{\mathbf{X}}}{F} G \boldsymbol{\Theta} \left(\bar{\mathbf{X}} - \mathbf{X}_t\right) \\
&\quad - \frac{1}{\gamma} \Phi_t^{-1} \frac{F_{\mathbf{X}}}{F} G \boldsymbol{\Sigma}_{\mathbf{X}}' \left(-\Phi_t \boldsymbol{\Lambda}_t\right) - (1-\alpha) \frac{1-\gamma}{\gamma} P_t^{-1} \frac{F_{\mathbf{X}}}{F} G \boldsymbol{\Sigma}_{\mathbf{X}}' \left(P_t \boldsymbol{\sigma}_P\right) \\
&\quad + (1-\alpha) \frac{1-\gamma}{\gamma^2} \Phi_t^{-1} P_t^{-1} G \left(P_t \boldsymbol{\sigma}_P\right)' \left(-\Phi_t \boldsymbol{\Lambda}_t\right) \\
&\quad + \frac{1}{2} \begin{pmatrix} \frac{1}{\gamma}\left(1 + \frac{1}{\gamma}\right) \Phi_t^{-2} G \left(-\Phi_t \boldsymbol{\Lambda}_t\right)' \left(-\Phi_t \boldsymbol{\Lambda}_t\right) \\ + (1-\alpha) \frac{1-\gamma}{\gamma} \left(1 + (1-\alpha) \frac{1-\gamma}{\gamma}\right) P_t^{-2} G \left(P_t \boldsymbol{\sigma}_P\right)' \left(P_t \boldsymbol{\sigma}_P\right) \\ + \operatorname{tr}\left(\frac{F_{\mathbf{X}\mathbf{X}}}{F} G \boldsymbol{\Sigma}_{\mathbf{X}}' \boldsymbol{\Sigma}_{\mathbf{X}}\right) \end{pmatrix}.
\end{aligned}$$

Simplifying:

$$\begin{aligned}
\frac{\mathcal{L}G}{G} &= \frac{1}{\gamma} R_t - (1-\alpha) \frac{1-\gamma}{\gamma} \pi_t + \frac{F_{\mathbf{X}}}{F} \boldsymbol{\Theta} \left(\bar{\mathbf{X}} - \mathbf{X}_t\right) \\
&\quad + \frac{1}{\gamma} \frac{F_{\mathbf{X}}}{F} \boldsymbol{\Sigma}_{\mathbf{X}}' \boldsymbol{\Lambda}_t - (1-\alpha) \frac{1-\gamma}{\gamma} \frac{F_{\mathbf{X}}}{F} \boldsymbol{\Sigma}_{\mathbf{X}}' \boldsymbol{\sigma}_P - (1-\alpha) \frac{1-\gamma}{\gamma^2} \boldsymbol{\sigma}_P' \boldsymbol{\Lambda}_t \\
&\quad + \frac{1}{2}\left(\frac{1}{\gamma}\left(1 + \frac{1}{\gamma}\right) \boldsymbol{\Lambda}_t' \boldsymbol{\Lambda}_t + (1-\alpha) \frac{1-\gamma}{\gamma} \left(1 + (1-\alpha) \frac{1-\gamma}{\gamma}\right) \boldsymbol{\sigma}_P' \boldsymbol{\sigma}_P + \operatorname{tr}\left(\frac{F_{\mathbf{X}\mathbf{X}}}{F} \boldsymbol{\Sigma}_{\mathbf{X}}' \boldsymbol{\Sigma}_{\mathbf{X}}\right)\right).
\end{aligned}$$

Substituting into (B.9):

$$\begin{aligned}
\left(\frac{1}{\gamma} \boldsymbol{\Lambda}_t' - (1-\alpha) \frac{1-\gamma}{\gamma} \boldsymbol{\sigma}_P' + \frac{F_{\mathbf{X}}}{F} \boldsymbol{\Sigma}_{\mathbf{X}}'\right) \boldsymbol{\Lambda}_t &= \frac{1}{\gamma} R_t - (1-\alpha) \frac{1-\gamma}{\gamma} \pi_t + \frac{F_{\mathbf{X}}}{F} \boldsymbol{\Theta} \left(\bar{\mathbf{X}} - \mathbf{X}_t\right) \\
&\quad + \frac{1}{\gamma} \frac{F_{\mathbf{X}}}{F} \boldsymbol{\Sigma}_{\mathbf{X}}' \boldsymbol{\Lambda}_t - (1-\alpha) \frac{1-\gamma}{\gamma} \frac{F_{\mathbf{X}}}{F} \boldsymbol{\Sigma}_{\mathbf{X}}' \boldsymbol{\sigma}_P \\
&\quad - (1-\alpha) \frac{1-\gamma}{\gamma^2} \boldsymbol{\sigma}_P' \boldsymbol{\Lambda}_t + \frac{1}{2\gamma}\left(1 + \frac{1}{\gamma}\right) \boldsymbol{\Lambda}_t' \boldsymbol{\Lambda}_t
\end{aligned}$$

$$
\begin{aligned}
&+(1-\alpha)\frac{1-\gamma}{2\gamma}\left(1+(1-\alpha)\frac{1-\gamma}{\gamma}\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P \\
&+\frac{1}{2}\mathrm{tr}\left(\frac{F_{\mathbf{XX}}}{F}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right)+\frac{G_t}{G}-R_t.
\end{aligned}
$$

Rearranging and collecting terms:

$$
\begin{aligned}
0 \quad = \quad & \left(\frac{1}{\gamma}-1\right)R_t-(1-\alpha)\frac{1-\gamma}{\gamma}\pi_t+\frac{F_{\mathbf{X}}}{F}\boldsymbol{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right) \\
&+\left(\frac{1}{\gamma}-1\right)\frac{F_{\mathbf{X}}}{F}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Lambda}_t-(1-\alpha)\frac{1-\gamma}{\gamma}\frac{F_{\mathbf{X}}}{F}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P \\
&+\left(1-\frac{1}{\gamma}\right)(1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_t \\
&+\frac{1}{2\gamma}\left(\frac{1}{\gamma}-1\right)\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t+(1-\alpha)\frac{1-\gamma}{2\gamma}\left(1+(1-\alpha)\frac{1-\gamma}{\gamma}\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P \\
&+\frac{1}{2}\mathrm{tr}\left(\frac{F_{\mathbf{XX}}}{F}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right)+\frac{F_t}{F}.
\end{aligned}
$$

It is at this point necessary to elaborate on market completeness. The optimal portfolio strategy is obtained equaling the diffusion terms of the dynamics of (log) wealth, written either as in (B.7) or as a linear combination of the dynamics of the underlying assets:

$$
\boldsymbol{\Sigma}\boldsymbol{\omega}=\frac{1}{\gamma}\boldsymbol{\Lambda}_t-(1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_P+\boldsymbol{\Sigma}_{\mathbf{X}}\frac{(F_{\mathbf{X}})'}{F}. \tag{B.10}
$$

This n-dimensional equation imposes that the N non-redundant traded assets, for which the volatility vectors are at the l.h.s., span the dynamics of optimal wealth at the r.h.s.. In the case of complete markets this happens without issues, as $\boldsymbol{\Sigma}$ is invertible. If instead markets are incomplete, we follow Sangvinatsos and Wachter (2005) and impose some further restrictions to make sure that the r.h.s. is completely spanned by the traded assets. We decompose the r.h.s. of the equation into a contribution spanned by the traded assets and a contribution orthogonal to the asset space. In order to decompose the vector of market prices of risk, we write $\boldsymbol{\Lambda}_t=\boldsymbol{\Lambda}_t^*+\boldsymbol{\nu}_t^*$, where $\boldsymbol{\Lambda}_t^*$ belongs to the column space of the volatility matrix of the traded assets and $\boldsymbol{\nu}_t^*$ belongs to the null space. We pre-multiply (B.10) by $\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'$:

$$
\underbrace{\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\Sigma}\boldsymbol{\omega}}_{0} \quad = \quad \frac{1}{\gamma}\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\left(\boldsymbol{\Lambda}_t^*+\boldsymbol{\nu}_t^*\right)
$$

$$
\begin{aligned}
& -(1-\alpha)\frac{1-\gamma}{\gamma}\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\sigma}_P \\
& +\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\Sigma}_{\mathbf{X}}\frac{\left(F_{\mathbf{X}}\right)'}{F}.
\end{aligned}
$$

This condition will affect the function F, which we still have to determine, and therefore the dynamics of optimal wealth and the optimal portfolio weights. As $\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\Lambda}_t^*=\mathbf{0}$, then:

$$
\boldsymbol{\nu}_t^*=(1-\alpha)(1-\gamma)\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\sigma}_P-\gamma\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\Sigma}_{\mathbf{X}}\frac{\left(F_{\mathbf{X}}\right)'}{F}.
$$

Pre-multiplying instead (B.10) by $\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'$:

$$
\begin{aligned}
\underbrace{\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\boldsymbol{\omega}}_{\boldsymbol{\Sigma}\boldsymbol{\omega}} &= \frac{1}{\gamma}\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\left(\boldsymbol{\Lambda}_t^*+\boldsymbol{\nu}_t^*\right)-(1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P \\
& +\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\frac{\left(F_{\mathbf{X}}\right)'}{F}.
\end{aligned}
$$

As $\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\nu}_t^*=\mathbf{0}$, by pre-multiplying again by $\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'$, we obtain the expression for the optimal portfolio weights (20).

We employ the following notations:

$$
\boldsymbol{\Lambda}_t^*=\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t=\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\left(\boldsymbol{\Lambda}_0+\boldsymbol{\Lambda}_1\mathbf{X}_t\right)=\boldsymbol{\Lambda}_0^*+\boldsymbol{\Lambda}_1^*\mathbf{X}_t,
$$

where:

$$
\begin{aligned}
\boldsymbol{\Lambda}_0^* &= \boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_0, \\
\boldsymbol{\Lambda}_1^* &= \boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_1,
\end{aligned}
$$

In the remaining, we also employ the following notations:

$$
\begin{aligned}
\boldsymbol{\sigma}_P^{\perp} &= \left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\sigma}_P, \\
\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp} &= \left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\Sigma}_{\mathbf{X}}.
\end{aligned}
$$

The market prices of risk can therefore be decomposed as:

$$\boldsymbol{\Lambda}_t = \boldsymbol{\Lambda}_t^* + \boldsymbol{\nu}_t^* = \boldsymbol{\Lambda}_0^* + \boldsymbol{\Lambda}_1^* \mathbf{X}_t + (1-\alpha)(1-\gamma)\boldsymbol{\sigma}_P^{\perp} - \gamma \boldsymbol{\Sigma}_{\mathbf{X}}^{\perp} \frac{(F_{\mathbf{X}})'}{F}.$$

We guess the functional form for $F(\mathbf{X}_t, t, T)$:

$$F(\mathbf{X}_t, \tau) = \exp\left\{\frac{1}{2}\mathbf{X}_t'\mathbf{B}_3(\tau)\mathbf{X}_t + \mathbf{B}_2(\tau)\mathbf{X}_t + B_1(\tau)\right\},$$

and compute the derivatives:

$$\begin{aligned}
F_{\mathbf{X}} &= \left(\frac{1}{2}\mathbf{X}_t'\left(\mathbf{B}_3(\tau) + \mathbf{B}_3'(\tau)\right) + \mathbf{B}_2(\tau)\right) F = \left(\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau) + \mathbf{B}_2(\tau)\right) F, \\
F_{\mathbf{XX}} &= \left(\frac{1}{2}\mathbf{X}_t'\left(\mathbf{B}_3(\tau) + \mathbf{B}_3'(\tau)\right) + \mathbf{B}_2(\tau)\right)' \left(\frac{1}{2}\mathbf{X}_t'\left(\mathbf{B}_3(\tau) + \mathbf{B}_3'(\tau)\right) + \mathbf{B}_2(\tau)\right) F \\
&\quad + \frac{1}{2}\left(\mathbf{B}_3(\tau) + \mathbf{B}_3'(\tau)\right) F \\
&= \frac{1}{4}\left(\mathbf{B}_3(\tau) + \mathbf{B}_3'(\tau)\right)' \mathbf{X}_t\mathbf{X}_t'\left(\mathbf{B}_3(\tau) + \mathbf{B}_3'(\tau)\right) F + \frac{1}{2}\mathbf{B}_2'(\tau)\mathbf{X}_t'\left(\mathbf{B}_3(\tau) + \mathbf{B}_3'(\tau)\right) F \\
&\quad + \frac{1}{2}\left(\mathbf{B}_3(\tau) + \mathbf{B}_3'(\tau)\right)' \mathbf{X}_t\mathbf{B}_2(\tau) F + \mathbf{B}_2'(\tau)\mathbf{B}_2(\tau) F + \frac{1}{2}\left(\mathbf{B}_3(\tau) + \mathbf{B}_3'(\tau)\right) F \\
&= \tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau) F + \mathbf{B}_2'(\tau)\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau) F + \tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t\mathbf{B}_2(\tau) F \\
&\quad + \mathbf{B}_2'(\tau)\mathbf{B}_2(\tau) F + \tilde{\mathbf{B}}_3(\tau) F, \\
F_t &= -\left(\frac{1}{2}\mathbf{X}_t'\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_3(\tau)\mathbf{X}_t + \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_2(\tau)\mathbf{X}_t + \frac{\mathrm{d}}{\mathrm{d}\tau}B_1(\tau)\right) F,
\end{aligned}$$

where $\tilde{\mathbf{B}}_3(\tau) = \frac{\mathbf{B}_3(\tau) + \mathbf{B}_3'(\tau)}{2}$. The market prices of risk can be rewritten as:

$$\begin{aligned}
\boldsymbol{\Lambda}_t &= \boldsymbol{\Lambda}_t^* + \boldsymbol{\nu}_t^* \\
&= \boldsymbol{\Lambda}_0^* + \boldsymbol{\Lambda}_1^*\mathbf{X}_t + (1-\alpha)(1-\gamma)\boldsymbol{\sigma}_P^{\perp} - \gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\left(\tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t + \mathbf{B}_2'(\tau)\right) \\
&= \boldsymbol{\Lambda}_0^* + (1-\alpha)(1-\gamma)\boldsymbol{\sigma}_P^{\perp} - \gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\mathbf{B}_2'(\tau) + \left(\boldsymbol{\Lambda}_1^* - \gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\tilde{\mathbf{B}}_3(\tau)\right)\mathbf{X}_t \\
&= \tilde{\boldsymbol{\Lambda}}_0^* + \tilde{\boldsymbol{\Lambda}}_1^*\mathbf{X}_t,
\end{aligned}$$

where $\tilde{\boldsymbol{\Lambda}}_0^* = \boldsymbol{\Lambda}_0^* + (1-\alpha)(1-\gamma)\boldsymbol{\sigma}_P^{\perp} - \gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\mathbf{B}_2'(\tau)$ and $\tilde{\boldsymbol{\Lambda}}_1^* = \boldsymbol{\Lambda}_1^* - \gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\tilde{\mathbf{B}}_3(\tau)$. Substituting in the

PDE for the expressions of R_t, π_t and $\mathbf{\Lambda}_t$, as well as for the derivatives of F:

$$\begin{aligned} 0 &= \left(\frac{1}{\gamma}-1\right)\left(R_0+\mathbf{R}_1'\mathbf{X}_t\right)-(1-\alpha)\frac{1-\gamma}{\gamma}\left(\pi_0+\boldsymbol{\pi}_1'\mathbf{X}_t\right)+\left(\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\mathbf{B}_2(\tau)\right)\mathbf{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right)\\ &+\left(\frac{1}{\gamma}-1\right)\left(\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\mathbf{B}_2(\tau)\right)\mathbf{\Sigma}_{\mathbf{X}}'\left(\tilde{\mathbf{\Lambda}}_0^*+\tilde{\mathbf{\Lambda}}_1^*\mathbf{X}_t\right)\\ &-(1-\alpha)\frac{1-\gamma}{\gamma}\left(\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\mathbf{B}_2(\tau)\right)\mathbf{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P+\left(1-\frac{1}{\gamma}\right)(1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_P'\left(\tilde{\mathbf{\Lambda}}_0^*+\tilde{\mathbf{\Lambda}}_1^*\mathbf{X}_t\right)\\ &+\frac{1}{2\gamma}\left(\frac{1}{\gamma}-1\right)\left(\tilde{\mathbf{\Lambda}}_0^*+\tilde{\mathbf{\Lambda}}_1^*\mathbf{X}_t\right)'\left(\tilde{\mathbf{\Lambda}}_0^*+\tilde{\mathbf{\Lambda}}_1^*\mathbf{X}_t\right)+(1-\alpha)\frac{1-\gamma}{2\gamma}\left(1+(1-\alpha)\frac{1-\gamma}{\gamma}\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P\\ &+\frac{1}{2}\mathrm{tr}\left(\left(\tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\mathbf{B}_2'(\tau)\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t\mathbf{B}_2(\tau)+\mathbf{B}_2'(\tau)\mathbf{B}_2(\tau)+\tilde{\mathbf{B}}_3(\tau)\right)\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\right)\\ &-\left(\frac{1}{2}\mathbf{X}_t'\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_3(\tau)\mathbf{X}_t+\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_2(\tau)\mathbf{X}_t+\frac{\mathrm{d}}{\mathrm{d}\tau}B_1(\tau)\right). \end{aligned}$$

Isolating the term in $\mathbf{X}_t' \ldots \mathbf{X}_t$ and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_3(\tau)$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_3(\tau) = -2\tilde{\mathbf{B}}_3(\tau)\mathbf{\Theta}+2\left(\frac{1}{\gamma}-1\right)\tilde{\mathbf{B}}_3(\tau)\mathbf{\Sigma}_{\mathbf{X}}'\tilde{\mathbf{\Lambda}}_1^*+\frac{1}{\gamma}\left(\frac{1}{\gamma}-1\right)\left(\tilde{\mathbf{\Lambda}}_1^*\right)'\tilde{\mathbf{\Lambda}}_1^*+\tilde{\mathbf{B}}_3(\tau)\mathbf{\Sigma}_X'\mathbf{\Sigma}_{\mathbf{X}}\tilde{\mathbf{B}}_3.$$

Isolating the term in $\mathbf{X}_t$ and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_2(\tau)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_2(\tau) &= \left(\frac{1}{\gamma}-1\right)\mathbf{R}_1'-(1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\pi}_1'+\bar{\mathbf{X}}'\mathbf{\Theta}'\tilde{\mathbf{B}}_3(\tau)-\mathbf{B}_2(\tau)\mathbf{\Theta}\\ &+\left(\frac{1}{\gamma}-1\right)\left(\tilde{\mathbf{\Lambda}}_0^*\right)'\mathbf{\Sigma}_{\mathbf{X}}\tilde{\mathbf{B}}_3(\tau)+\left(\frac{1}{\gamma}-1\right)\mathbf{B}_2(\tau)\mathbf{\Sigma}_{\mathbf{X}}'\tilde{\mathbf{\Lambda}}_1^*\\ &-(1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_P'\mathbf{\Sigma}_{\mathbf{X}}\tilde{\mathbf{B}}_3(\tau)-(1-\alpha)\left(\frac{\gamma-1}{\gamma}\right)^2\boldsymbol{\sigma}_P'\tilde{\mathbf{\Lambda}}_1^*\\ &+\frac{1}{\gamma}\left(\frac{1}{\gamma}-1\right)\left(\tilde{\mathbf{\Lambda}}_0^*\right)'\tilde{\mathbf{\Lambda}}_1^*+\mathbf{B}_2(\tau)\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\tilde{\mathbf{B}}_3(\tau). \end{aligned}$$

Isolating the homogeneous term and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}B_1(\tau)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}B_1(\tau) &= \left(\frac{1}{\gamma}-1\right)R_0-(1-\alpha)\frac{1-\gamma}{\gamma}\pi_0+\mathbf{B}_2(\tau)\mathbf{\Theta}\bar{\mathbf{X}}+\left(\frac{1}{\gamma}-1\right)\mathbf{B}_2(\tau)\mathbf{\Sigma}_{\mathbf{X}}'\tilde{\mathbf{\Lambda}}_0^*\\ &-(1-\alpha)\frac{1-\gamma}{\gamma}\mathbf{B}_2(\tau)\mathbf{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P-(1-\alpha)\left(\frac{\gamma-1}{\gamma}\right)^2\boldsymbol{\sigma}_P'\tilde{\mathbf{\Lambda}}_0^*\\ &+\frac{1}{2\gamma}\left(\frac{1}{\gamma}-1\right)\left(\tilde{\mathbf{\Lambda}}_0^*\right)'\tilde{\mathbf{\Lambda}}_0^*+(1-\alpha)\frac{1-\gamma}{2\gamma}\left(1+(1-\alpha)\frac{1-\gamma}{\gamma}\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P\\ &+\frac{1}{2}\mathbf{B}_2(\tau)\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\mathbf{B}_2'(\tau)+\frac{1}{2}\mathrm{tr}\left(\tilde{\mathbf{B}}_3(\tau)\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\right). \end{aligned}$$

In order to write the optimal portfolio weights, remember that $\frac{F_{\mathbf{X}}}{F} = \mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\mathbf{B}_2(\tau)$. Then, substituting into (20):

$$\boldsymbol{\omega}_t = \frac{1}{\gamma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t - (1-\alpha)\frac{1-\gamma}{\gamma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P + \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t + \mathbf{B}_2'(\tau)\right). \tag{B.11}$$

B.2 Optimal portfolio strategy for an infinitely risk-averse individual and constant market prices of risk

For $\gamma \to \infty$, in the case of constant market prices of risk ($\boldsymbol{\Lambda}_1 = \mathbf{0}$), it is possible to write an explicit solution to the problem. Indeed, in this case $\mathbf{B}_3(\tau) \to \mathbf{0}$ and the optimal portfolio strategy becomes independent of the current state $\mathbf{X}_t$:

$$\boldsymbol{\omega}_t = (1-\alpha)\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P + \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{B}_2'(\tau), \tag{B.12}$$

where:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_2(\tau) = -\mathbf{R}_1' + (1-\alpha)\boldsymbol{\pi}_1' - \mathbf{B}_2(\tau)\boldsymbol{\Theta},$$

for which the solution is:

$$\mathbf{B}_2(\tau) = \left(\mathbf{R}_1' - (1-\alpha)\boldsymbol{\pi}_1'\right)\left(\mathbf{e}^{-\boldsymbol{\Theta}\tau} - \mathbf{I}\right)\boldsymbol{\Theta}^{-1}.$$

We can then rewrite the optimal portfolio strategy as:

$$\boldsymbol{\omega}_t = (1-\alpha)\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P + \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\left[\mathbf{R}_1'\left(\mathbf{e}^{-\boldsymbol{\Theta}\tau} - \mathbf{I}\right)\boldsymbol{\Theta}^{-1}\right]' - (1-\alpha)\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\left[\boldsymbol{\pi}_1'\left(\mathbf{e}^{-\boldsymbol{\Theta}\tau} - \mathbf{I}\right)\boldsymbol{\Theta}^{-1}\right]'. \tag{B.13}$$

Remember that, for constant risk premia, the coefficient $\mathbf{A}_1^N$, used for nominal bond pricing, satisfies the following relation:

$$\frac{\partial}{\partial\tau}\mathbf{A}_1^N(\tau) = -\mathbf{A}_1^N(\tau)\boldsymbol{\Theta} - \mathbf{R}_1', \tag{B.14}$$

the solution for $\mathbf{A}_1^N$ is:

$$\mathbf{A}_1^N(\tau) = \mathbf{R}_1' \left(\mathbf{e}^{-\boldsymbol{\Theta}\tau} - \mathbf{I}\right) \boldsymbol{\Theta}^{-1}. \tag{B.15}$$

The same applies to the coefficient $\mathbf{A}_1^I$, used for real bond pricing:

$$\frac{\partial}{\partial \tau}\mathbf{A}_1^I(\tau) = -\mathbf{A}_1^I(\tau)\,\boldsymbol{\Theta} - \left(\mathbf{R}_\mathbf{1}' - \boldsymbol{\pi}_1'\right), \tag{B.16}$$

for which the solution is:

$$\mathbf{A}_1^I(\tau) = \left(\mathbf{R}_\mathbf{1}' - \boldsymbol{\pi}_1'\right)\left(\mathbf{e}^{-\boldsymbol{\Theta}\tau} - \mathbf{I}\right)\boldsymbol{\Theta}^{-1}.$$

Therefore:

$$\boldsymbol{\omega}_t = (1-\alpha)\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\left(\boldsymbol{\Sigma}_\mathbf{X}\mathbf{A}_1^I(\tau) + \boldsymbol{\sigma}_P\right) + \alpha\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_\mathbf{X}\mathbf{A}_1^N(\tau),$$

where the first term, with weight $1-\alpha$, replicates an inflation-indexed bond with time-to-maturity τ, and the second term, with weight α, replicates a nominal bond with time-to-maturity τ.

B.3 Utility from suboptimal strategies

Given a certain degree of money illusion α, the optimal portfolio strategy is given by (B.11). We are interested in assessing the expected utility of an investor, following a certain portfolio strategy, which is suboptimal with respect to her degree of money illusion $\hat{\alpha}$. In particular, we are interested in assessing the utility loss sustained by a non-illusioned agent ($\hat{\alpha} = 0$), in the case where she follows the same portfolio strategy as an illusioned individual ($\alpha > 0$). The portfolio weights, calculated (sub-optimally) for $\alpha > 0$, take the form:

$$\begin{aligned}\hat{\boldsymbol{\omega}}_t(\tau) &= \frac{1}{\gamma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\left(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1\mathbf{X}_t\right) + (1-\alpha)\left(1-\frac{1}{\gamma}\right)\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P \\ &\quad + \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_\mathbf{X}\left(\tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t + \mathbf{B}_2'(\tau)\right) \\ &\equiv \hat{\boldsymbol{\omega}}_0(\tau) + \hat{\boldsymbol{\omega}}_1(\tau)\mathbf{X}_t,\end{aligned} \tag{B.17}$$

where $\tau = T - t$ and:

$$\begin{aligned}
\hat{\boldsymbol{\omega}}_0(\tau) &= \frac{1}{\gamma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_0 + (1-\alpha)\left(1-\frac{1}{\gamma}\right)\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P + \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{B}_2'(\tau),\\
\hat{\boldsymbol{\omega}}_1(\tau) &= \frac{1}{\gamma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_1 + \left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\mathbf{B}}_3(\tau).
\end{aligned}$$

The expected utility over terminal wealth (evaluated for $\hat{\alpha}$), $\hat{J}(t, W_t, \mathbf{X}_t, P_t)$, is a martingale, as it represent an expectation of future utility. Thus, it satisfies the following PDE:

$$\frac{\partial}{\partial t}\hat{J} + \mathcal{L}\hat{J} = 0,$$

where $\mathcal{L}\hat{J}$ is the following differential operator:

$$\begin{aligned}
\mathcal{L}\hat{J} &= \hat{J}_W W_t\left(R_t + \hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t\right) + \hat{J}_{\mathbf{X}}\boldsymbol{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) + \hat{J}_P P_t \pi_t\\
&+\hat{J}_{\mathbf{X}W} W_t \boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_t(\tau) + \hat{J}_{WP} W_t P_t \hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P + \hat{J}_{\mathbf{X}P} P_t \boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P\\
&+\frac{1}{2}\hat{J}_{WW} W_t^2 \hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_t(\tau) + \frac{1}{2}\mathrm{tr}\left(\hat{J}_{\mathbf{X}\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right) + \frac{1}{2}\hat{J}_{PP} P_t^2 \boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P.
\end{aligned}$$

We guess the following functional form for $\hat{J}(t, W_t, \mathbf{X}_t, P_t)$:

$$\hat{J}(t, W_t, \mathbf{X}_t, P_t) = \frac{W_t^{1-\gamma} P_t^{-(1-\hat{\alpha})(1-\gamma)}}{1-\gamma} H(\mathbf{X}_t, t, T).$$

Substituting into the PDE and simplifying:

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}H + (1-\gamma)\left(R_t + \hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t\right)H + H_{\mathbf{X}}\boldsymbol{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) - (1-\hat{\alpha})(1-\gamma)\pi_t H\\
&+ (1-\gamma) H_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_t(\tau) - (1-\hat{\alpha})(1-\gamma)^2\hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P H - (1-\hat{\alpha})(1-\gamma) H_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P\\
&-\frac{1}{2}\gamma(1-\gamma)\hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_t(\tau) H + \frac{1}{2}\mathrm{tr}\left(H_{\mathbf{X}\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right)\\
&+\frac{1}{2}(1-\hat{\alpha})(1-\gamma)(2-\hat{\alpha}-\gamma+\alpha\gamma)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P H.
\end{aligned}$$

We make a guess on the functional form of H:

$$H(\mathbf{X}_t, \tau) = \exp\left\{\frac{1}{2}\mathbf{X}_t'\hat{\mathbf{B}}_3(\tau)\mathbf{X}_t + \hat{\mathbf{B}}_2(\tau)\mathbf{X}_t + \hat{B}_1(\tau)\right\}.$$

Writing the partial derivatives:

$$H_{\mathbf{X}} = \left(\frac{1}{2}\mathbf{X}_t'\left(\hat{\mathbf{B}}_3(\tau)+\hat{\mathbf{B}}_3'(\tau)\right)+\hat{\mathbf{B}}_2(\tau)\right)H = \left(\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau)+\hat{\mathbf{B}}_2(\tau)\right)H,$$

$$\begin{aligned} H_{\mathbf{XX}} &= \frac{1}{4}\left(\hat{\mathbf{B}}_3(\tau)+\hat{\mathbf{B}}_3'(\tau)\right)'\mathbf{X}_t\mathbf{X}_t'\left(\hat{\mathbf{B}}_3(\tau)+\hat{\mathbf{B}}_3'(\tau)\right)H+\frac{1}{2}\hat{\mathbf{B}}_2'(\tau)\mathbf{X}_t'\left(\hat{\mathbf{B}}_3(\tau)+\hat{\mathbf{B}}_3'(\tau)\right)H \\ &\quad +\frac{1}{2}\left(\hat{\mathbf{B}}_3(\tau)+\hat{\mathbf{B}}_3'(\tau)\right)'\mathbf{X}_t\hat{\mathbf{B}}_2(\tau)H+\hat{\mathbf{B}}_2'(\tau)\hat{\mathbf{B}}_2(\tau)H+\frac{1}{2}\left(\hat{\mathbf{B}}_3(\tau)+\hat{\mathbf{B}}_3'(\tau)\right)H \\ &= \left(\tilde{\hat{\mathbf{B}}}_3(\tau)\mathbf{X}_t\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau)+\hat{\mathbf{B}}_2'(\tau)\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau)+\tilde{\hat{\mathbf{B}}}_3(\tau)\mathbf{X}_t\hat{\mathbf{B}}_2(\tau)+\hat{\mathbf{B}}_2'(\tau)\hat{\mathbf{B}}_2(\tau)+\tilde{\hat{\mathbf{B}}}_3(\tau)\right)H, \end{aligned}$$

$$\frac{\partial}{\partial t}H = -\left(\frac{1}{2}\mathbf{X}_t'\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_3(\tau)\mathbf{X}_t+\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_2(\tau)\mathbf{X}_t+\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{B}_1(\tau)\right)H,$$

where $\tilde{\hat{\mathbf{B}}}_3(\tau) = \frac{\hat{\mathbf{B}}_3(\tau)+\hat{\mathbf{B}}_3'(\tau)}{2}$. Substituting into the PDE the partial derivatives, as well as the quantities R_t, π_t, $\mathbf{\Lambda}_t$ (note that $\mathbf{\Sigma}'\mathbf{\Lambda}_t = \mathbf{\Sigma}'\mathbf{\Lambda}_t^*$) and $\hat{\boldsymbol{\omega}}_t(\tau)$:

$$\begin{aligned} 0 &= -\left(\frac{1}{2}\mathbf{X}_t'\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_3(\tau)\mathbf{X}_t+\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_2(\tau)\mathbf{X}_t+\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{B}_1(\tau)\right) \\ &\quad (1-\gamma)\left(R_0+\mathbf{R}_1'\mathbf{X}_t+(\hat{\boldsymbol{\omega}}_0(\tau)+\hat{\boldsymbol{\omega}}_1(\tau)\mathbf{X}_t)'\mathbf{\Sigma}'(\mathbf{\Lambda}_0+\mathbf{\Lambda}_1\mathbf{X}_t)\right) \\ &\quad +\left(\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau)+\hat{\mathbf{B}}_2(\tau)\right)\mathbf{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right) \\ &\quad -(1-\hat{\alpha})(1-\gamma)(\pi_0+\boldsymbol{\pi}_1'\mathbf{X}_t) \\ &\quad +(1-\gamma)\left(\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau)+\hat{\mathbf{B}}_2(\tau)\right)\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}(\hat{\boldsymbol{\omega}}_0(\tau)+\hat{\boldsymbol{\omega}}_1(\tau)\mathbf{X}_t) \\ &\quad -(1-\hat{\alpha})(1-\gamma)^2(\hat{\boldsymbol{\omega}}_0(\tau)+\hat{\boldsymbol{\omega}}_1(\tau)\mathbf{X}_t)'\mathbf{\Sigma}'\boldsymbol{\sigma}_P \\ &\quad -(1-\hat{\alpha})(1-\gamma)\left(\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau)+\hat{\mathbf{B}}_2(\tau)\right)\mathbf{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P \\ &\quad -\frac{1}{2}\gamma(1-\gamma)(\hat{\boldsymbol{\omega}}_0(\tau)+\hat{\boldsymbol{\omega}}_1(\tau)\mathbf{X}_t)'\mathbf{\Sigma}'\mathbf{\Sigma}(\hat{\boldsymbol{\omega}}_0(\tau)+\hat{\boldsymbol{\omega}}_1(\tau)\mathbf{X}_t) \\ &\quad +\frac{1}{2}\mathrm{tr}\left(\left(\tilde{\hat{\mathbf{B}}}_3(\tau)\mathbf{X}_t\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau)+\hat{\mathbf{B}}_2'(\tau)\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau)+\tilde{\hat{\mathbf{B}}}_3(\tau)\mathbf{X}_t\hat{\mathbf{B}}_2(\tau)+\hat{\mathbf{B}}_2'(\tau)\hat{\mathbf{B}}_2(\tau)+\tilde{\hat{\mathbf{B}}}_3(\tau)\right)\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\right) \\ &\quad +\frac{1}{2}(1-\hat{\alpha})(1-\gamma)(2-\hat{\alpha}-\gamma+\hat{\alpha}\gamma)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P. \end{aligned}$$

Isolating the terms in $\mathbf{X}_t'\ldots\mathbf{X}_t$ and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_3(\tau)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_3(\tau) &= 2(1-\gamma)\hat{\boldsymbol{\omega}}_1'(\tau)\mathbf{\Sigma}'\mathbf{\Lambda}_1-2\tilde{\hat{\mathbf{B}}}_3(\tau)\mathbf{\Theta}+2(1-\gamma)\tilde{\hat{\mathbf{B}}}_3(\tau)\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}\hat{\boldsymbol{\omega}}_1(\tau) \\ &\quad -\gamma(1-\gamma)\hat{\boldsymbol{\omega}}_1'(\tau)\mathbf{\Sigma}'\mathbf{\Sigma}\hat{\boldsymbol{\omega}}_1(\tau)+\tilde{\hat{\mathbf{B}}}_3(\tau)\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\tilde{\hat{\mathbf{B}}}_3(\tau). \end{aligned}$$

Isolating the terms in $\mathbf{X}_t$ and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_2(\tau)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_2(\tau) &= (1-\gamma)\left(\mathbf{R}_1' + \hat{\boldsymbol{\omega}}_0'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_1 + \boldsymbol{\Lambda}_0'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_1(\tau)\right) \\
&\quad + \bar{\mathbf{X}}'\boldsymbol{\Theta}'\tilde{\hat{\mathbf{B}}}_3(\tau) - \hat{\mathbf{B}}_2(\tau)\boldsymbol{\Theta} - (1-\hat{\alpha})(1-\gamma)\boldsymbol{\pi}_1' \\
&\quad + (1-\gamma)\hat{\boldsymbol{\omega}}_0'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\hat{\mathbf{B}}}_3(\tau) + (1-\gamma)\hat{\mathbf{B}}_2(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_1(\tau) \\
&\quad - (1-\hat{\alpha})(1-\gamma)^2\boldsymbol{\sigma}_P'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_1(\tau) - (1-\hat{\alpha})(1-\gamma)\boldsymbol{\sigma}_P'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\hat{\mathbf{B}}}_3(\tau) \\
&\quad - \gamma(1-\gamma)\hat{\boldsymbol{\omega}}_0'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_1(\tau) + \hat{\mathbf{B}}_2(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\hat{\mathbf{B}}}_3(\tau).
\end{aligned}$$

Isolating the homogeneous terms and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{B}_1(\tau)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{B}_1(\tau) &= (1-\gamma)\left(R_0 + \hat{\boldsymbol{\omega}}_0'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_0\right) + \hat{\mathbf{B}}_2(\tau)\boldsymbol{\Theta}\bar{\mathbf{X}} - (1-\hat{\alpha})(1-\gamma)\pi_0 \\
&\quad + (1-\gamma)\hat{\mathbf{B}}_2(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_0(\tau) - (1-\hat{\alpha})(1-\gamma)^2\hat{\boldsymbol{\omega}}_0'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P \\
&\quad - (1-\hat{\alpha})(1-\gamma)\hat{\mathbf{B}}_2(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P - \frac{1}{2}\gamma(1-\gamma)\hat{\boldsymbol{\omega}}_0'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_0(\tau) \\
&\quad + \frac{1}{2}\hat{\mathbf{B}}_2(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\hat{\mathbf{B}}_2'(\tau) + \frac{1}{2}\mathrm{tr}\left(\tilde{\hat{\mathbf{B}}}_3(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right) \\
&\quad + \frac{1}{2}(1-\hat{\alpha})(1-\gamma)(2-\hat{\alpha}-\gamma+\hat{\alpha}\gamma)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P.
\end{aligned}$$

B.4 Certainty equivalent utility loss

The certainty equivalent utility loss ℓ, incurred by following a suboptimal strategy, can be measured in terms of fraction of initial wealth. It can be obtained by solving the following problem:

$$\hat{J}(0, W_0, \mathbf{X}_0, P_0) = J(0, W_0(1-\ell), \mathbf{X}_0, P_0).$$

Developing the r.h.s.:

$$\hat{J}(0, W_0, \mathbf{X}_0, P_0) = \frac{W_0^{1-\gamma}(1-\ell)^{1-\gamma}P_0^{-(1-\alpha)(1-\gamma)}}{1-\gamma}\left[F(\mathbf{X}_0, 0, T)\right]^{\gamma} = (1-\ell)^{1-\gamma}J(0, W_0, \mathbf{X}_0, P_0),$$

and, finally, the initial certainty equivalent loss can be expressed as:

$$\ell = 1 - \left(\frac{\hat{J}(0, W_0, \mathbf{X}_0, P_0)}{J(0, W_0, \mathbf{X}_0, P_0)}\right)^{\frac{1}{1-\gamma}}.$$

It is useful to calculate an annualized certainty equivalent loss, which, for an investment horizon equal to τ, we evaluate as:

$$\ell_{\text{ann}} = 1 - \left(\frac{\hat{J}\left(0, W_0, \mathbf{X}_0, P_0\right)}{J\left(0, W_0, \mathbf{X}_0, P_0\right)} \right)^{\frac{1}{(1-\gamma)\tau}} = 1 - (1-\ell)^{\frac{1}{\tau}} .$$

C Empirical findings imposing a zero unexpected inflation risk premium

The estimation of the inflation risk premium is an object of debate in the literature concerning inflation-indexed securities. In our setting, in particular, the unexpected inflation risk premium is equal to $\sigma_P' \mathbf{\Lambda}_t$ and determines the difference between the expected inflation π_t and the break-even inflation:

$$\text{bei}_t = R_t - r_t = \pi_t - \sigma_P' \mathbf{\Lambda}_t. \tag{C.18}$$

We are interested in assessing whether our empirical findings in terms of portfolio strategy and welfare are robust with respect to the estimate obtained for the unexpected inflation risk premium. The base case estimate of this quantity, for the setting with constant risk premia, is non-negligible and equal to $\sigma_P' \tilde{\mathbf{\Lambda}} = 1.17\%$. We decided to repeat our analysis imposing this quantity to be equal to zero. In order to do so, we needed to determine a modified vector of constant market prices of risk, $\tilde{\mathbf{\Lambda}}_0$, so that $\sigma_P' \mathbf{\Lambda}_0 = 0$. $\tilde{\mathbf{\Lambda}}$ is a 5×1 vector. Therefore, we needed to impose four additional constraints to pin it down. We chose to impose that the risk premia on the 2- and 10-year nominal bonds, as well as that of the 10-year inflation-indexed bond and the stock, are unchanged with respect to their base case values. This translates into the following conditions:

$$\begin{bmatrix} \mathbf{A}_1^N(2)\,\mathbf{\Sigma}_X' \\ \mathbf{A}_1^B(10)\,\mathbf{\Sigma}_X' \\ \mathbf{A}_1^I(10)\,\mathbf{\Sigma}_X' + \boldsymbol{\sigma}_P' \\ \boldsymbol{\sigma}_P' \\ \boldsymbol{\sigma}_S' \end{bmatrix} \tilde{\mathbf{\Lambda}}_0 = \begin{bmatrix} \mathbf{A}_1^N(2)\,\mathbf{\Sigma}_X' \mathbf{\Lambda}_0 \\ \mathbf{A}_1^B(10)\,\mathbf{\Sigma}_X' \mathbf{\Lambda}_0 \\ \left(\mathbf{A}_1^I(10)\,\mathbf{\Sigma}_X' + \boldsymbol{\sigma}_P'\right)\mathbf{\Lambda}_0 \\ 0 \\ \boldsymbol{\sigma}_S' \mathbf{\Lambda}_0 \end{bmatrix}.$$

Figure A.1 shows the optimal portfolio strategy for different degrees of money illusion, as well as the certainty equivalent annualized loss attributable to money illusion. The results are virtually identical to those obtained for the base case estimate of $\mathbf{\Lambda}_0$ (Figure 6).

[Figure A.1 about here.]

Figure A.2 instead shows the optimal portfolio positions when either the inflation-indexed bond or the nominal bond are removed from the investable universe, as well as the perceived loss by the investors with different values of α. The results are, again, very similar to those shown in Figure 7.

[Figure A.2 about here.]

Overall, it seems that the conclusions we draw in Section 4 are not sensitive to the estimate of the realized inflation risk premium.

D Empirical findings with unconstrained volatility of time-varying risk premia

We report in this section the empirical results obtained where we do not impose any restriction on the volatility of the risk premia during the estimation phase.

D.1 Estimation

We consider a specification with time-varying risk premia, initially with no restrictions imposed on the matrix $\mathbf{\Lambda}_1$. As in Christensen et al. (2010), we then iterate the estimation procedure, by progressively imposing a zero restriction on the element of $\mathbf{\Lambda}_1$ with the lowest t-statistics. We stop when all the elements of $\mathbf{\Lambda}_1$ have a t-stat higher than 2. The parameter estimates are listed in Table A.1

[Table A.1 about here.]

The summary statistics and characteristics of the distributions of the most relevant economic and financial quantities, both historical and as implied by the estimated parameters, are listed in Table A.2. It is very interesting to note how the model-implied volatility of expected inflation is affected by considering time-varying risk premia with no restriction on their volatility. In addition, the estimated risk premia seem to be extremely (and unrealistically) volatile in this case.

[Table A.2 about here.]

Table A.3 shows the correlations both from the historical distribution and as implied by the estimated parameters (for both variable and constant risk premia).

[Table A.3 about here.]

Figure A.3 shows the time series of the model-implied macroeconomic variables, the risk premia and the maximum achievable Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond and the stock index). When compared to Figure 2b, it is noticeable how the volatility of the expected inflation (which is unobservable) is higher in this setting. The annualized bond risk premia range from -5% to 20%, while the equity premium ranges from -20% to 40%. Their volatilities also seem to be excessively high. The time series of mean-variance portfolio weights is even more surprising: for a myopic investor with $\gamma = 10$, the portfolio weights of the bonds range from -600% to 900%. The ex-ante Sharpe ratio (in annual terms) achievable with the mean-variance portfolio is on average higher than 1, with peaks that reach 5.

[Figure A.3 about here.]

D.2 Portfolio strategy and welfare

Portfolio strategy Figure A.4 shows the optimal portfolio strategy when investors have access to the full investment universe and the state variables are at their long-run means. For short investment horizons, the optimal strategy is similar to the case where risk premia are constant (Figure 6) or time-varying with a constrained volatility (Figure 8). When the horizon is increased, the impact of the time-varying risk premia with unconstrained volatility significantly affects the portfolio strategy. However, the qualitative results at the long-run mean of the state of the economy are similar to the other specifications, with an increase of the nominal bond position and a decrease of the real bond position with α, as well as positions in the stock and cash, which are rather insensitive to α.

[Figure A.4 about here.]

Utility loss due to money illusion The welfare analysis in the graph at the bottom, showing the certainty equivalent loss attributable to money illusion, confirms that the welfare effect of money illusion is substantial. The pattern of the loss is very similar to the case of constant risk premia, although the size of the relative loss seems to be lower than in the cases of constant or volatility-constrained risk premia. For example, the annualized loss is about 0.7% for a totally illusioned investor with a 30-year horizon w.r.t. a rational investor (rather than 1.2% for the case with volatility-constrained risk premia).

Figure A.4, however, shows relative losses, which somehow hide the actual value of certainty equivalent returns. In-sample certainty equivalent returns are, in this specification, unrealistically high. For example, a non-illusioned agent with a 30-year horizon has a certainty equivalent real annualized return equal to 4.36% when risk premia are constant, and equal to 4.50% when risk premia are time-varying with a constrained volatility. The in-sample certainty equivalent real annualized return is instead unrealistically equal to 21.27% when risk premia are time-varying with an unconstrained volatility.

This figure is due to the fact that optimal portfolio positions, as soon as the state of the economy departs from the long-run mean, quickly assume unrealistically high values. To point out this issue, we show in Figure A.5 the time series of in-sample portfolio weights, considering a non-illusioned investor ($\alpha = 0$) with a 10-year horizon, for the three specifications of risk premia. The weights are calculated for each date considering the time series of the state variables $\mathbf{X}_t$ and the estimated model parameters. As can be noticed, the positions, which are expressed in fractions of total wealth, are constant for the case of constant risk premia, time-varying within a realistically implementable range (from -100% to 200%) for the case of volatility-constrained risk premia, and time-varying within an unrealistically wide range (from -1500% to 2000%) for the case of unconstrained risk premia. In particular, the in-sample overfitting of bond risk premia causes incredibly large and offsetting positions between the nominal bond and the real bond, which correspond to quasi-arbitrage opportunities. The extremely large in-sample certainty equivalents are a consequence of this same issue.

[Figure A.5 about here.]

Perceived utility loss due to the unavailability of inflation-indexed bonds for different degrees of money illusion We consider now the case where one of the bonds is removed from the investable universe. In particular, Figure A.6a shows the case where the inflation-indexed bond is unavailable. In terms of portfolio positions, qualitatively, the effect of money illusion on the optimal allocation is similar to the case with volatility-constrained risk premia (Figure 9a), although in this case the effect of money illusion on the portfolio positions seems to be less pronounced.

[Figure A.6 about here.]

We then evaluate the opportunity cost, perceived by investors with different levels of α, of excluding the real bond. As can be noticed in Figure A.6a, the opportunity cost of excluding the inflation-indexed bond, although slightly higher for the non-illusioned investor, is huge for any level of α. In particular, the annualized loss reaches a level of about 8% per annum for a 30-year horizon. This is a consequence of the fact that, once a bond is excluded from the investable universe, it is not possible for the investor to exploit the quasi-arbitrage opportunities arising from the in-sample over-fitted bond risk premia. This result is consistent with empirical results obtained by other studies relying on unconstrained essentially affine term structure models, such as the huge utility losses incurred when following suboptimal strategies, as documented by Sangvinatsos and Wachter (2005) and Barillas (2011), or the extremely high sensitivity to parameter uncertainty, as documented by Feldhütter et al. (2012).

Finally, Figure 9 shows the complementary case, where the nominal bond is excluded from the investable universe. As expected, the annualized loss is slightly higher for the money-illusioned investor, but the value of about 7.5% per annum, applicable to all investors, seems to be unrealistically high.

Table A.1: Parameter estimates when no constraints on the volatility of risk premia are imposed.

The table shows the maximum-likelihood estimates of the model parameters. The values in parentheses are the standard errors of the estimates. Starting from an unrestricted matrix $\mathbf{\Lambda}_1$, the estimation is iterated by progressively imposing a zero restriction on the element of $\mathbf{\Lambda}_1$ with the lowest t-statistics. The results reported in the table correspond to the estimate obtained when all the elements of $\mathbf{\Lambda}_1$ have a t-stat higher than 2. The sample period runs from January 1999 until January 2016.

R_0	$\mathbf{R}_1$	π_0	$\boldsymbol{\pi}_1$	$\mathbf{\Theta}$			σ_ϵ^B	σ_ϵ^I
0.0185	0.3627	0.0206	0.0692	0.0647	0.7241	1.0434	0.0012	0.0007
(0.0001)	(0.0013)	(0.0021)	(0.0050)	(0.0032)	(0.3668)	(0.0371)	(0.0000)	(0.0000)
	−0.4186		−0.6482	0.0216	0.6808	−0.3107		
	(0.0053)		(0.0957)	(0.0019)	(0.1412)	(0.0210)		
	0.3255		−1.9020	−0.0224	0.1209	2.2387		
	(0.0135)		(0.4692)	(0.0018)	(0.1825)	(0.4280)		
$\mathbf{\Lambda}_0$	$\mathbf{\Lambda}_1$			$\mathbf{\Sigma}_X$			$\boldsymbol{\sigma}_P$	$\boldsymbol{\sigma}_S$
−0.6897	0	−62.1901	0	0.0205	0.0069	−0.0050	−0.0001	0.0420
(0.0338)		(17.7928)		(0.0010)	(0.0009)	(0.0007)	(0.0007)	(0.0105)
0.4655	0	0	0	0	0.0123	0.0021	−0.0008	−0.0358
(0.0518)					(0.0006)	(0.0007)	(0.0007)	(0.0108)
−0.2056	0	−38.2214	−224.6685	0	0	0.0088	−0.0043	−0.0426
(0.0613)		(18.3875)	(46.5592)			(0.0004)	(0.0006)	(0.0101)
1.2118	0	0	−210.3079	0	0	0	0.0087	−0.0155
(0.2718)			(50.0411)				(0.0004)	(0.0095)
0.7563	−9.0400	0	0	0	0	0	0	0.1399
(0.2746)	(4.5026)							(0.0069)

Table A.2: Historical and model-implied summary statistics when no constraints on the volatility of risk premia are imposed.

The table shows annualized historical and model-implied means and volatilities of bond yields, equity log returns, realized inflation and their model-implied means and volatilities. The table also shows the model-implied means and volatilities of the bond risk premia, the equity risk premium, the nominal risk-free rate and the expected inflation.

Time series	Mean value		Volatility	
	Estimation	Data	Estimation	Data
3M nominal yield	1.91%	1.87%	0.59%	0.64%
6M nominal yield	1.97%	1.97%	0.59%	0.61%
1Y nominal yield	2.10%	2.15%	0.61%	0.74%
2Y nominal yield	2.37%	2.38%	0.70%	0.85%
3Y nominal yield	2.64%	2.63%	0.81%	0.91%
5Y nominal yield	3.14%	3.12%	0.95%	0.96%
7Y nominal yield	3.55%	3.54%	0.98%	0.96%
10Y nominal yield	3.99%	4.01%	0.93%	0.95%
5Y real yield	1.29%	1.28%	0.92%	0.97%
7Y real yield	1.51%	1.54%	0.88%	0.86%
10Y real yield	1.81%	1.80%	0.84%	0.77%
Log realized inflation	2.06%	2.17%	0.97%	1.07%
Equity log returns	5.63%	4.98%	15.71%	15.72%
3M nominal risk premium	0.12%		0.16%	
6M nominal risk premium	0.25%		0.29%	
1Y nominal risk premium	0.52%		0.49%	
2Y nominal risk premium	1.07%		1.18%	
3Y nominal risk premium	1.62%		2.33%	
5Y nominal risk premium	2.55%		5.07%	
7Y nominal risk premium	3.21%		7.42%	
10Y nominal risk premium	3.73%		9.52%	
5Y real risk premium	2.19%		5.44%	
7Y real risk premium	2.70%		6.12%	
10Y real risk premium	3.25%		7.10%	
Realized inflation risk premium	1.11%		0.93%	
Equity risk premium	5.02%		14.95%	
Nominal risk-free rate	1.85%		0.61%	
Expected inflation	2.06%		2.16%	

Table A.3: Correlations between asset returns and economic variables when no constraints on the volatility of risk premia are imposed.

Panel (a) shows unconditional correlations between nominal and real bond returns, stock returns and realized inflation, calculated from the monthly time series. Panel (b) reports one-month conditional pairwise correlations between nominal and real bond returns, stock returns, realized inflation, nominal interest rate, expected inflation and real interest rate.

(a) Data

	3M nom	1Y nom	2Y nom	5Y nom	10Y nom	5Y real	10Y real	Equity	CPI
3M nom	1.000								
1Y nom	0.658	1.000							
2Y nom	0.471	0.924	1.000						
5Y nom	0.257	0.708	0.889	1.000					
10Y nom	0.108	0.499	0.669	0.904	1.000				
5Y real	−0.000	0.311	0.409	0.501	0.476	1.000			
10Y real	−0.009	0.322	0.440	0.623	0.682	0.910	1.000		
Equity	−0.160	−0.299	−0.352	−0.323	−0.253	0.038	0.000	1.000	
CPI	−0.124	−0.150	−0.139	−0.190	−0.234	0.350	0.156	0.070	1.000

(b) Time-varying risk premia without volatility restrictions

	3M nom	1Y nom	2Y nom	5Y nom	10Y nom	5Y real	10Y real	Equity	CPI	R	π	r
3M nom	1.000											
1Y nom	0.929	1.000										
2Y nom	0.754	0.942	1.000									
5Y nom	0.412	0.712	0.905	1.000								
10Y nom	0.212	0.538	0.780	0.968	1.000							
5Y real	0.087	0.214	0.327	0.483	0.616	1.000						
10Y real	0.057	0.273	0.458	0.670	0.798	0.962	1.000					
Equity	−0.233	−0.313	−0.344	−0.306	−0.235	0.090	0.004	1.000				
CPI	0.057	−0.049	−0.132	−0.179	−0.140	0.286	0.161	0.067	1.000			
R	−0.974	−0.852	−0.631	−0.255	−0.060	−0.070	0.009	0.173	−0.128	1.000		
π	−0.327	−0.500	−0.580	−0.521	−0.359	0.494	0.272	0.390	0.459	0.188	1.000	
r	0.036	0.131	0.181	0.136	−0.004	−0.788	−0.597	−0.293	−0.477	0.063	−0.904	1.000

Figure A.1: Optimal portfolio strategy for $\gamma = 10$. Utility loss with respect to a non-illusioned investor. Constant risk premia and zero realized inflation risk premium.

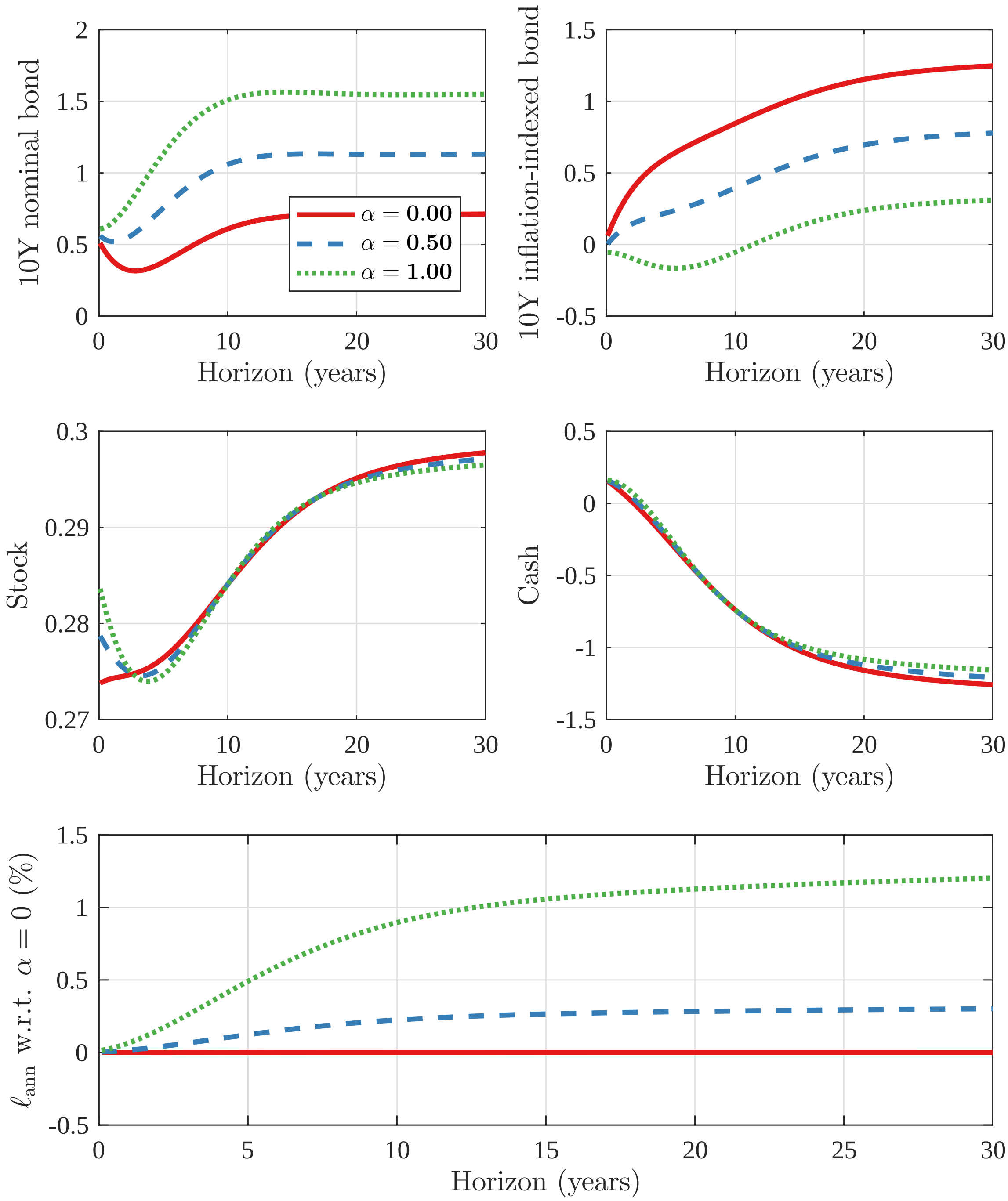
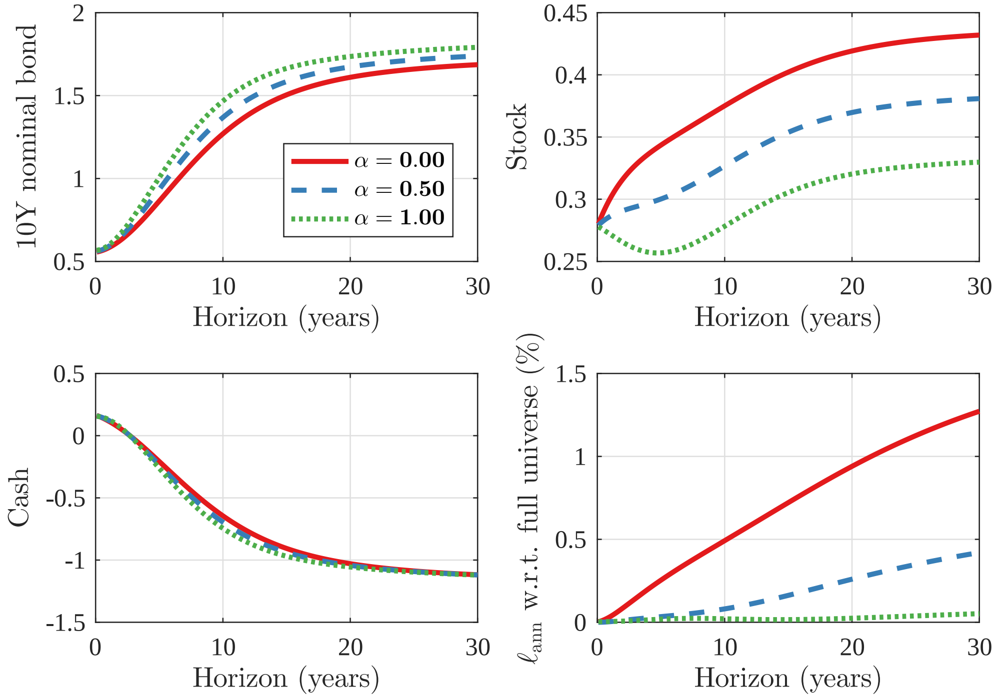


Figure A.2: Optimal portfolio strategy for $\gamma = 10$ when the 10-year inflation-indexed or nominal bond is excluded from the investable universe. Utility loss with respect to the case where the full investable universe is available. Constant risk premia and zero realized inflation risk premium.

(a) Inflation-indexed bond removed from investable universe



(b) Nominal bond removed from investable universe

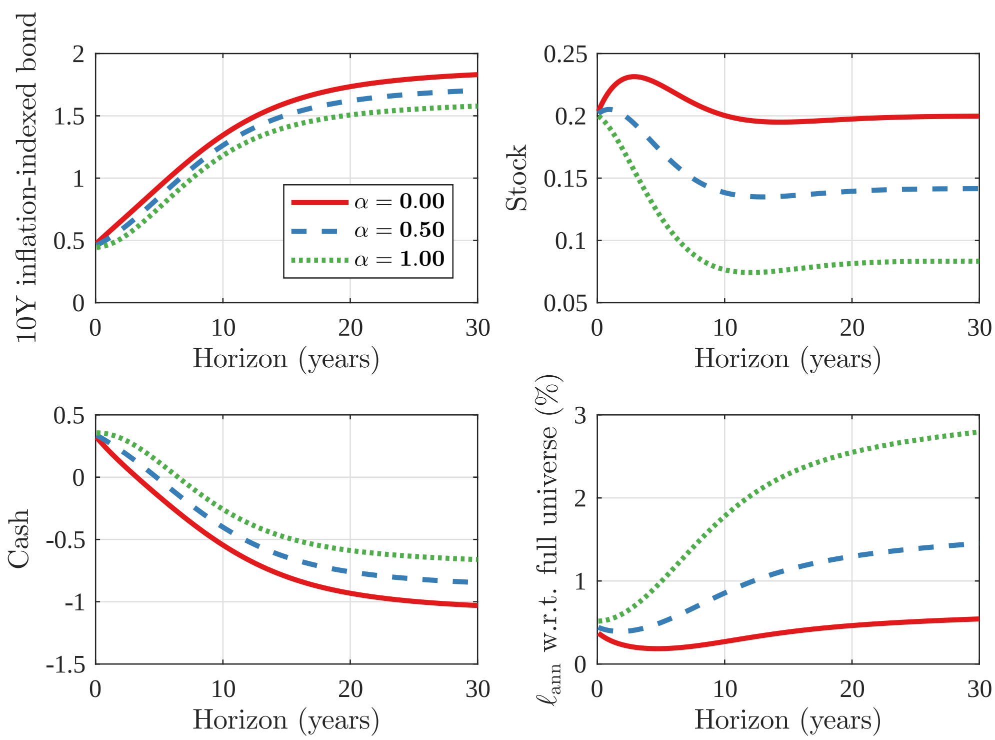


Figure A.3: Time series of macroeconomic variables, risk premia, mean-variance portfolio positions and the maximum achievable Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond, the stock index and $\gamma = 10$). Time-varying risk premia (unconstrained volatility).

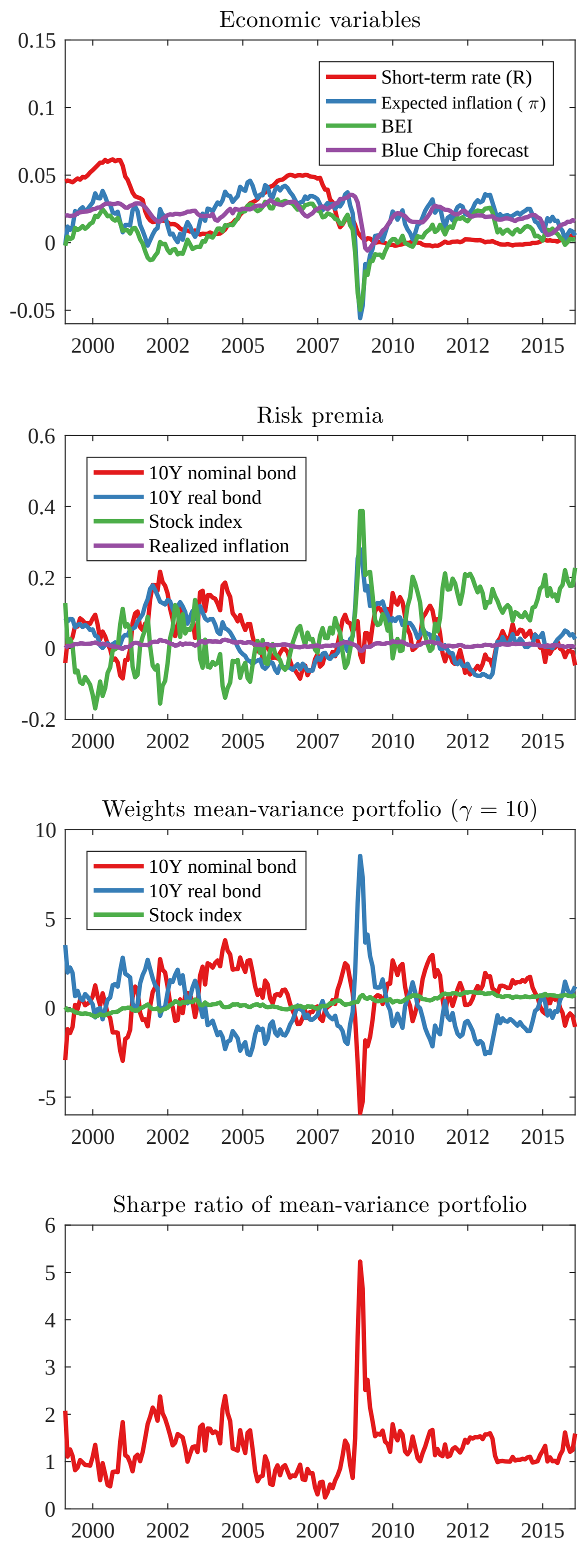


Figure A.4: Optimal portfolio strategy for $\gamma = 10$. Utility loss with respect to a non-illusioned investor. Time-varying risk premia (unconstrained volatility).

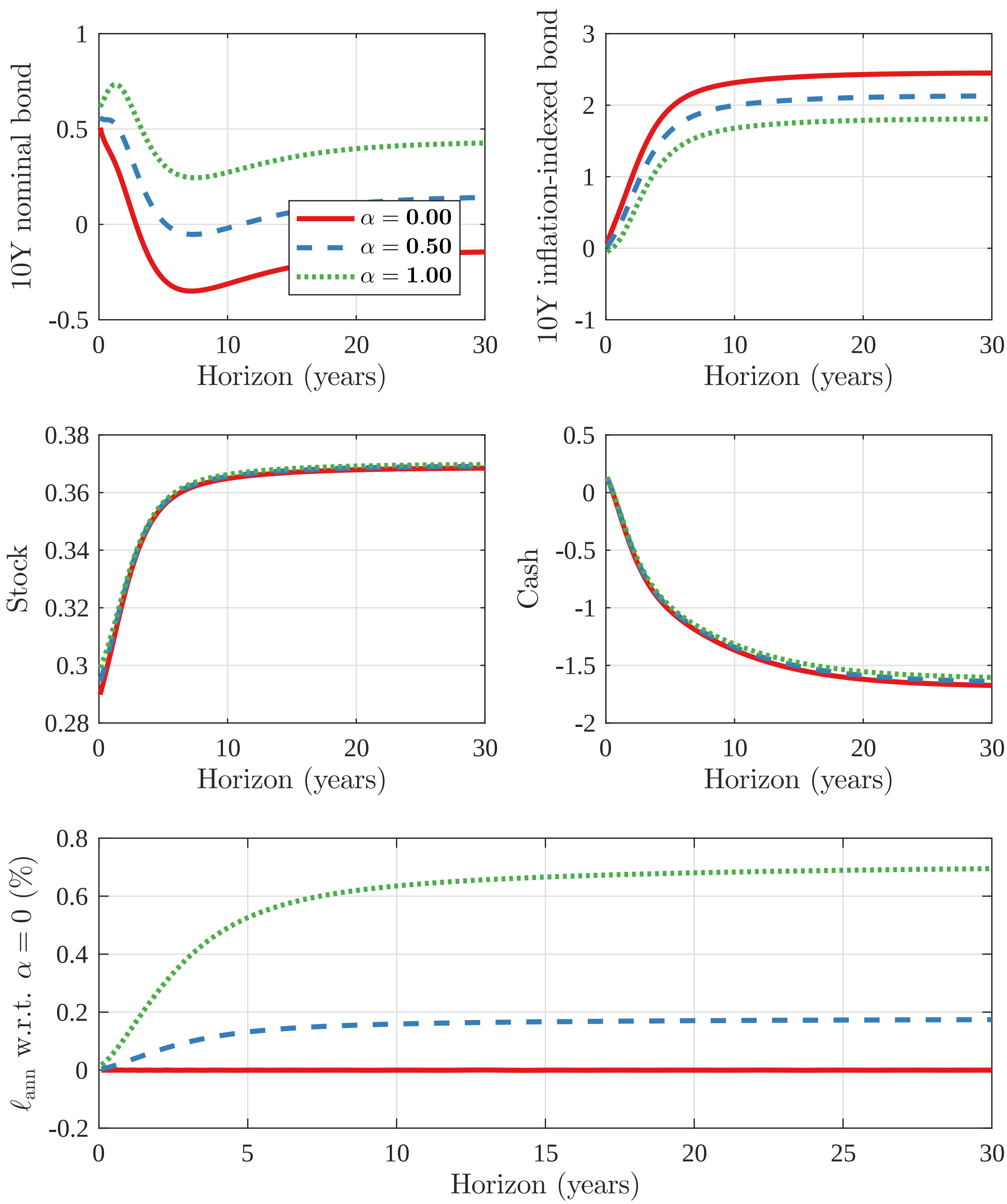
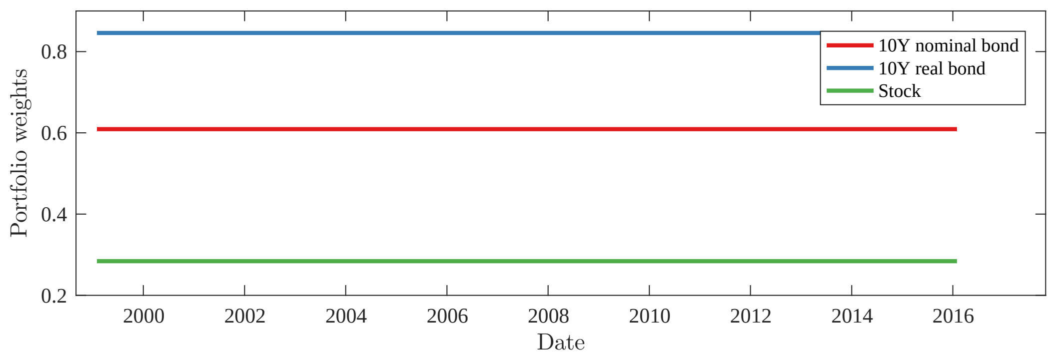


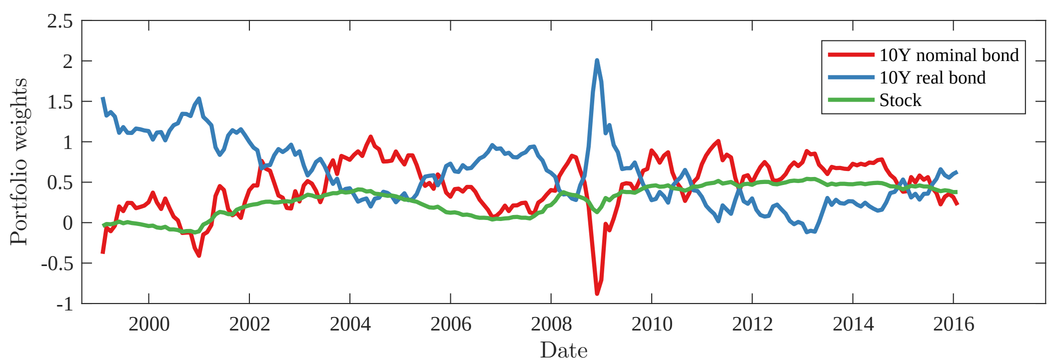
Figure A.5: Time series of in-sample optimal portfolio weights for a non-illusioned investor, obtained for different risk premia specifications.

The figures show the optimal in-sample portfolio weights, calculated for each date considering the time series of the state variables $\mathbf{X}_t$ and the estimated model parameters. The investor is not money-illusioned, i.e. $\alpha = 0$, and the investment horizon is 10-year long. Panel (a) shows the optimal portfolio weights obtained considering the model specification with constant risk premia, using the estimates in Table 1a. Panel (b) shows the optimal portfolio weights obtained considering the model specification with time-varying risk premia constrained in volatility, using the estimates in Table 1b. Panel (c) shows the optimal portfolio weights obtained considering the model specification with time-varying risk premia with no volatility constraints, using the estimates in Table A.1. All values are expressed as fractions of wealth (e.g. a weight equal to 1.5 means 150% of total wealth).

(a) Constant risk premia



(b) Time-varying risk premia (constrained volatility)



(c) Time-varying risk premia (unconstrained volatility)

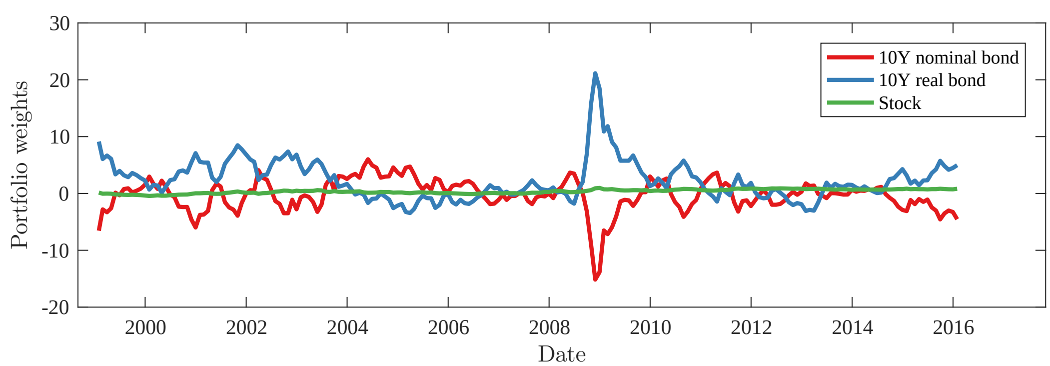
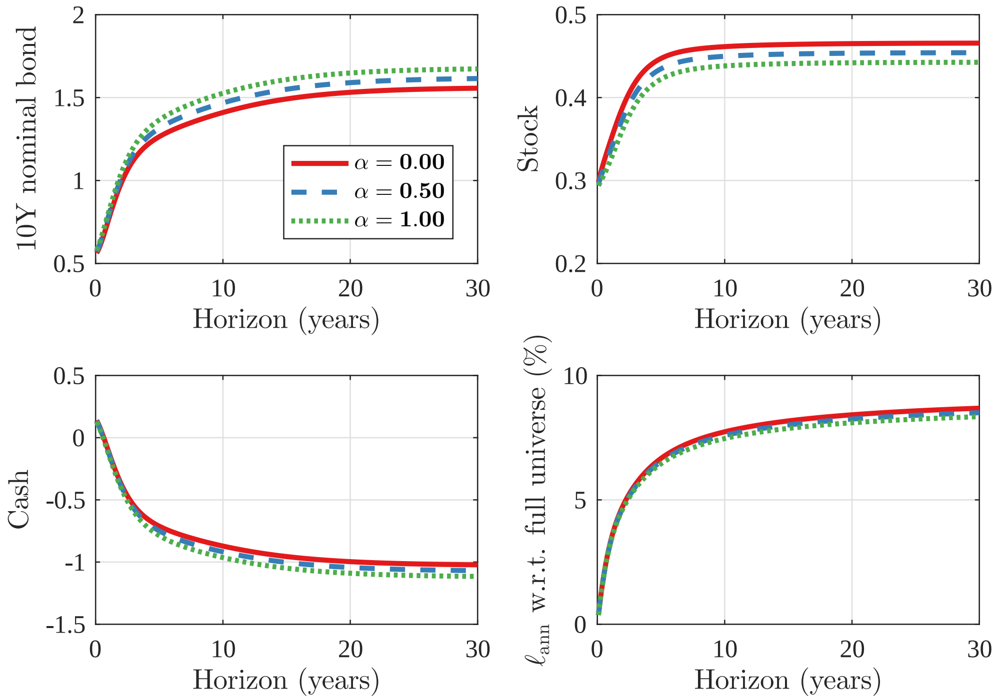


Figure A.6: Optimal portfolio strategy for $\gamma = 10$ when the 10-year inflation-indexed or nominal bond is excluded from the investable universe. Utility loss with respect to the case where the full investable universe is available. Time-varying risk premia (unconstrained volatility).

(a) Inflation-indexed bond removed from investable universe



(b) Nominal bond removed from investable universe