Risk Sharing and the Term Structure of Interest Rates*

Andrés Schneider (UCLA)§

JOB MARKET PAPER

(click for latest version)

November 24, 2017

Abstract

I propose a general equilibrium model with heterogeneous investors to explain the key properties of the U.S. real and nominal term structure of interest rates. I find that differences in investors' willingness to substitute consumption across time are critical to account for nominal and real yields dynamics. When the endogenous amount of credit supplied by risk-tolerant investors is low, the aggregate price of risk and the real interest rate are high. Thus, real bonds are risky. I study nominal bonds under both exogenous and endogenous (Taylor rule) inflation. I find that when the Taylor loading on inflation is greater than one, the nominal term structure is upward sloping regardless of the correlation between nominal and real shocks. I use the model to shed light on two salient interest rate puzzles: (1) the secular decline of long-term real and nominal rates since the 1980s, and (2) the sudden spike in real yields at the height of the Great Recession.

JEL classification: G11, G12, G20, E43, E44.

Keywords: Yield Curve, Risk Sharing, Credit Market, Elasticity of Intertemporal Substitution.

*I am deeply grateful to my committee, Andy Atkeson, Mike Chernov, Pierre-Olivier Weill, and Stavros Panageas, for their invaluable guidance and support. I also thank Saki Bigio, YiLi Chien, Sebastian Di Tella, François Geerolf, Gary Hansen, Mahyar Kargar, Lars Lochstoer, Francis Longstaff, Rody Manuelli, Maarten Meeuwis, Tyler Muir, Lee Ohanian, Emilio Osambela, Francisco Palomino, Gainluca Rinaldi, Juan Sanchez, Alejandro Van der Ghote, Irina Zviadadze, and participants at the UCLA Macro Proseminar, UCSB Graduate Workshop, Anderson Student Seminar, The Central Bank of Argentina Seminar, Federal Reserve Board MFMA Seminar, Federal Reserve Board Financial Stability Seminar, and the Federal Reserve Bank of St. Louis Seminar for useful suggestions and comments. Any errors are my own.

§Correspondence: Department of Economics, UCLA, Bunche Hall 8283, Los Angeles, CA, 90095. Email: anschneider@ucla.edu. Website: http://www.andresschneider.com

1 Introduction

Long-term nominal yields on U.S. government bonds display a higher mean and lower volatility than short-term yields. Data from inflation-protected bonds (TIPS) show that the real yield curve exhibits similar patterns, which suggests that real risks are important for understanding the nominal term structure. Characterizing the macroeconomic fundamentals that drive these common features in the real and nominal yield curve, in a unified framework, has been a long-standing challenge for macroeconomists and financial economists (Gürkaynak and Wright, 2012). The main contribution of this paper is to propose a model in which the credit market plays a key role in understanding these salient properties of U.S. real and nominal yield curves.

Indeed, in theory and practice, interest rates are determined in the credit market, which renders it a natural starting point for the study of term structure dynamics. The basic feature of the credit market is that heterogeneous investors lend to and borrow from each other with the purpose of sharing risks; heterogeneity creates gains from trade. I incorporate this idea in a general equilibrium term structure model and find that the difference in investors' willingness to substitute consumption across time is critical in capturing the properties of the nominal and real yield curves we observe in the data.

In this economy, the quantity of credit generates endogenous fluctuations in asset prices. In particular, term premia and yields are endogenously time-varying due to fluctuations in a single state variable that summarizes the credit conditions in the economy: the market value of leveraged investors' net worth over the total market value of net worth. This state variable has been underscored by many macro models that feature a credit market (with and without frictions), but this paper is the first to explicitly examine its influence on the term structure of interest rates.

The economic mechanism hinges on two assumptions. First, I motivate a credit market by assuming that investors have different attitudes toward risks (Dumas, 1989; Wang, 1996; Chan and Kogan, 2002; Bhamra and Uppal, 2009; Longstaff and Wang, 2012; Gârleanu and Panageas, 2015; Barro, Fernández-Villaverde, Levintal and Mollerus, 2017; Hall, 2017). Thus, in equilibrium, risk-tolerant investors issue short-term debt to finance leveraged positions in risky assets, which implies that their net worth is relatively more exposed to aggregate shocks. As a consequence, the effect of exogenous i.i.d. shocks on asset prices is persistently amplified by risk-tolerant investors' net worth, which generates endogenous fluctuations in the term structure. Second, I assume that investors with a high risk aversion (RA) coefficient exhibit a smaller elasticity of intertemporal substitution (EIS) than that implied by time-additive constant relative risk-aversion preferences. This assumption is key in capturing the quantitative properties of the term structure, because agents with relatively low EIS must be compensated with higher interest rates in equilibrium. Recursive preferences are essential to accommodate this feature.

The main mechanism is as follows. A negative aggregate shock generates a contraction in leveraged risk-tolerant investors' net worth, reducing their aggregate ability to supply credit. The contraction in aggregate credit produces an increase in the price of credit—i.e., the spot real rate. This is because a more risk-averse investor, who has a low EIS, must be incentivized to reallocate his portfolio and smooth consumption over time. In addition, the price of risk rises endogenously, because a more risk-averse investor is at the margin in the market for risky assets. The increase in the real rate implies that real bond prices become low in bad states: The marginal investor requires a positive premium to hold real bonds.

This relationship between interest rates and the credit market delivers an average upward-sloping real term structure. On average, fixing investors' expectations about future short-term rates, long-term yields are higher than short-term yields. This is because long-term bonds command a higher term premium; the larger the horizon of a bond, the more likely it will drop in value during bad states, because they have higher exposure to variations in interest rates. Put differently, long-term bonds have a higher elasticity with respect to endogenous changes in the share of net worth held by risk-tolerant investors (the model's endogenous state variable, which summarizes the amount of credit in the economy). I evaluate this elasticity in the empirical section of the paper.

A further implication of the mechanism is that it takes time for the credit market to recompose after a negative shock. Simply put, a contraction in aggregate credit implies lower asset prices, which further implies that risk-tolerant investors can supply less credit. This persistence shows up in equilibrium asset prices, and in particular in long-term bonds: The longer the horizon of the bond, the larger the effect of the credit market's persistence. This translates into a higher volatility of long-term bond prices relative to short-term bonds. However, since long-term bonds are stationary, this volatility grows at a slower pace than the horizon of bonds.[1] As a result, since yields are (log) bond prices divided by the horizon of the bond, long-term yields are always less volatile than short-term yields.

After reviewing the main theoretical underpinnings of the mechanism described above, I study the nominal term structure of interest rates. For this, I consider two alternative inflation processes: exogenous and endogenous (derived via a Taylor rule).

The purpose of introducing exogenous inflation is to study a decomposition between the real and nominal components of the nominal term premium. In this analysis, the nominal component is driven by the exogenous negative correlation between cash flow and inflation shocks (e.g., Cox, Ingersoll and Ross, 1985; Wachter, 2006; Piazzesi and Schneider, 2006; Bansal and Shaliastovich, 2013). That is, if inflation occurs in bad states, the marginal investor requires a premium to hold nominal bonds. The real component is driven by the endogenous risk generated in the credit market. In this decomposition, I find that even with a large negative correlation between inflation and real shocks, the real component explains 80% of the average nominal term premium observed in the data. This result is in line with recent studies showing the importance of the real component in the nominal term structure (e.g., Abrahams, Adrian, Crump, Moench and Yu, 2016).

Motivated by this result, I derive a nominal term structure that is purely driven by the real component. Using a Taylor rule, I derive an endogenous inflation process that is consistent with both the policy rule and the marginal investor's nominal pricing kernel (e.g., Gallmeyer, Hollifield, Palomino and Zin, 2007). As a result, inflation does not introduce new shocks, as in the exogenous case—i.e., the nominal term premium is not driven by nominal risk. I obtain an average slope of the nominal term structure that is in line with the data, driven by the fact that the Taylor loading on the policy rule is greater than one.

[1] I obtain an invariant distribution in the economy by using a simple OLG framework based on Blanchard (1985) and Gârleanu and Panageas (2015). I review this in detail in Section 3.

Sensitivity analysis shows that the larger (smaller) the Taylor loading, the smaller (larger) the mean and volatility of inflation, and the flatter (steeper) the nominal yield curve.

I next evaluate the central theoretical predictions of the model. For this, I first extract aggregate shocks from macroeconomic data. I exploit the fact that I consider an aggregate endowment with i.i.d. growth rates, and therefore aggregate shocks are straightforward to identify (under the null of my model). Second, I feed the shocks into the model to study the predictions for the endogenous state variables. In this step, I compare fluctuations in the amount of credit in the model against fluctuations in the data (total credit to private sector over GDP), and I find the model captures these fluctuations relatively well. After checking the predictions for credit, I compute the implied series for the endogenous state variable in the model, and use those series to check whether the model's key predictions are verified in the term structure data.

In particular, I regress yields from the data onto the model's endogenous state variable (derived after feeding the macro shocks). The purpose of this is to test the main model's predictions: the sensitivity of both yields and slope (difference in yields) with respect to the endogenous state variable. The model predicts that long-term yields are less sensitive to the endogenous state variable than short-term yields (i.e., they are less volatile), and that the slope is positive and nonlinearly related to the endogenous state variable. Regressions using actual data for yields and the model's implied series for the state variable confirm these two central predictions.

I then study whether the endogenous state variable can capture the fluctuations in the short-term nominal interest rate. This is the key prediction of the endogenous inflation case. I find that the endogenous state variable can account for a significant portion of short-term nominal interest rate variability, even after controlling for other well-studied macro factors since Ang and Piazzesi (2003).

After validating the theoretical predictions, I provide an application of the model's mechanism to shed light on two puzzles regarding yields (Campbell, Shiller and Viceira, 2009). These are: (1) the sudden spike in the level and the reversion of the slope of the real term structure at the height of the Great Recession; and (2) the secular decline of nominal and real rates over the last 30 years. The objective is not only to provide further evidence on the mechanism I propose, but also to show that the connection between the credit market and the term structure provides a coherent perspective for important macroeconomic phenomena.

Specifically, in both applications I stress the role of the aggregate EIS. The sudden spike in real rates during the Great Recession can be rationalized as a sudden collapse in credit that produced a drastic reduction in aggregate willingness to substitute consumption into the future. The secular decline in nominal and real rates can be rationalized by the observed contemporaneous increase in the amount of credit in the economy, which in the model translates into a decrease in the price of credit (i.e., the spot risk-free rate). Due to the single factor structure of the model, the decrease in short-term rates is also reflected in long-term rates. In addition, the model implies a secular decrease in inflation expectations pinned down by the Taylor policy rule—which is consistent with survey data, as shown by Chernov and Mueller (2012), among others.

I conclude by comparing the model's prediction for the state variable with an alternative interpretation. Prior literature has interpreted risk-tolerant investors as the owners of financial institutions, or "credit suppliers" (e.g., Longstaff and Wang, 2012; Silva, 2016; Santos and Veronesi, 2016; Drechsler, Savov and Schnabl, 2017). In this view, the net worth of financial firms should be useful in capturing the credit conditions in the economy, and therefore yield dynamics. Following this alternative view, I construct the ratio of the market value of financial firms over the total market value of firms. I report the time series of this measure and compare it with those of the endogenous state variable in the model. In find the correlation of these two variables is significantly positive.

Literature. My paper fits into three strands of literature: heterogeneous agents and the credit market, macro-finance models of the term structure, and empirical literature studies that show the importance of credit measures in capturing yields dynamics.

First, my paper is related to recent papers in macroeconomics and finance that stress the role of the credit market in determining the behavior of equilibrium asset prices. A common theme in these papers is that agents exhibit heterogeneous exposure to aggregate risks (i.e., a group of agents operates with leverage in equilibrium), driven by differences in a technological feature (preferences, productivity, menu of assets, beliefs, information, etc.). Within this strand, my work is in line with many studies that focus on the positive implications for asset prices and macroeconomic quantities with a frictionless credit market (e.g., Dumas, 1989; Wang, 1996; Chan and Kogan, 2002; Bhamra and Uppal, 2009; Longstaff and Wang, 2012; Gârleanu and Panageas, 2015; Barro et al., 2017; Hall, 2017; Schneider, 2017).

Specifically, Longstaff and Wang (2012) study an endowment economy in which agents feature heterogeneous constant relative risk-aversion preferences and analyze the role of the credit market on asset prices. In particular, they find that real yields on perpetual bonds are smaller than the short-term real yield (i.e., a downward-sloping real yield curve). Following this line, Hall (2017) studies an economy with differences in risk aversion and argues that the secular decline in the average real rates can be explained by an increase in the wealth share of risk-averse agents. An implicit result in this analysis is that real bonds are hedges, and therefore the yields on long-term real bonds have a lower mean than short-term yields. Gârleanu and Panageas (2015), extend the analysis to heterogeneous agents with recursive preferences, in which the economy has a simple OLG structure to obtain a stationary wealth distribution, and underscore the importance of heterogeneous preferences in determining the equity premium; Barro et al. (2017) studies an economy in which heterogeneous agents share aggregate risk in an economy subject to disasters and focus on the implications for the supply of safe assets; Wang (1996) considers an economy with heterogeneous agents with constant relative risk aversion and studies the theoretical properties of real yields; Schneider (2017) studies an economy in which fluctuations in premiums are driven by the interaction between endogenous changes in balance sheets and exogenous changes in macro volatility.

Relative to this first strand of literature, in this paper I show that the credit market is a key macroeconomic fundamental to understand the real and nominal term structure in a unified framework. In my results, I highlight the role of differences in EIS. In fact, when investors exhibit the same EIS—but different RA—the economy exhibits a downward-sloping real term structure with only 10% of the yields' volatility we observe in the data.

A second strand this paper is related to is the macro-finance models of the term structure with a representative agent. This literature is extensive, but leading examples are Piazzesi and Schneider (2006), who study a Long Run Risk economy, where the representative agent exhibits very high risk aversion, and dislikes exogenous inflation such that more than compensates the downward sloping real yield curve; Bansal and Shaliastovich (2013), who study a Long Run Risk economy with stochastic volatility and analyze the implications for interest rates and currencies; Relative to this second strand of literature, in this paper I focus on the role of the credit market in determining the properties of the real and nominal term structure. This connection cannot be made in a representative agent setup.

Within the representative agent literature, Wachter (2006) introduces an exogenous time variation in the habits framework of Campbell and Cochrane (1999) and finds an upward-sloping nominal and real term structure of yields but also of their corresponding volatilities (the 5-year yield is more volatile than the 1-year). In the data, however, the longer the maturity of the bond (either nominal or real), the smaller the volatility of the yield. Also, Wachter's model predicts a slope of the nominal and real yield curves that are very similar. In the data I report below, also documented by Backus, Boyarchenko and Chernov (2017), the slope of the nominal term structure is at least twice the slope of the real term structure—the nominal term structure is steeper than the real. In my paper, in addition to providing an economic mechanism that links the term structure to credit market activity, I show that my model can also capture the fact that long-term yields are less volatile than short-term, and also that the nominal term structure is steeper than the real. Indeed, I show that the slope of the nominal term structure vis-à-vis the real can be rationalized by the reaction of monetary policy to the endogenous risks generated in the credit market.

Several papers have introduced further structure to the representative agent framework, and they study the term structure in a large scale dynamic stochastic general equilibrium model (DSGE) with production. Prominent examples are Rudebusch and Swanson (2008, 2012). The mechanism I propose in this paper, which generates endogenous time variation in the aggregate RA and EIS, can be introduced in a reduced form in such large scale DSGE models.

Lastly, my paper is related to empirical papers that stress the role of macro variables associated with the credit market in driving term premia over the business cycle. Haddad and Sraer (2015) use a measure of banks' exposure to interest rates ("income gap") to capture the key properties of term premia, using a partial equilibrium model to illustrate the mechanism. Greenwood and Vayanos (2014) show empirically how the supply and the maturity structure of government bonds affect bond yields and expected returns. Relative to these papers, the state variable in my paper can be interpreted as a macro factor that is helpful in capturing the yield's dynamics.

2 Preliminary Evidence: U.S. Nominal and Real Yield Curves

This section documents the salient properties of the U.S. real and nominal yield curves described above.[2] In the quantitative part of the model, I seek to match this evidence. I elaborate on the evidence for both nominal and real yields, for different maturities, in three different samples. I report this data, at quarterly

[2] In the appendix I report evidence for the U.K. Although that evidence is not the empirical objective of this paper, I report that both the nominal and real yield curves in the U.K. have similar properties: In the full sample, they are inversely U-shaped (yield curves are upward sloping but revert after 10-year maturity), and the volatility of long-term yields is slightly smaller than the volatility of short-term yields.

frequency, in Table 1. The source for nominal rates is Gürkaynak, Sack and Wright (2007). For real rates, I consider TIPS from Gürkaynak, Sack and Wright (2010) for the period 2003:Q1 onward, and I incorporate real rates from Chernov and Mueller (2012) for the period 1971:Q3-2002:Q4.

The full sample (reported in Panel A) is from 1971:Q3 to 2016:Q4. This is the same sample documented in Backus et al. (2017), but includes 2015 and 2016. I then consider two subsamples. The first (Short sample I, in Panel B) includes only the period reported by Gürkaynak et al. (2010), 2003:Q3-2016:Q4, so it excludes the data from Chernov and Mueller (2012). The second, (Short sample II, in Panel C), excludes the period of the Great Recession (1971:Q1 to 2008:Q2). That is, I exclude a time with massive policy interventions. In particular, the short-term nominal rate was set up to zero.

The main conclusion from the evidence, in all subsamples, is that both the real and the nominal term structure share similar properties: Long-term yields have higher means and smaller volatilities than short-term yields. Indeed, in Short sample I, the volatility levels of both real and nominal term structures are almost the same. It is worth emphasizing that the average slope of the nominal term structure is approximately twice the slope of the real term structure, which raises a question about additional sources of risks captured by the nominal yield curve.

3 Model

In this section I study an endowment economy populated by heterogeneous investors, and I assume that the sole source of heterogeneity among investors is in their preferences. In particular, investors differ in their RA and EIS. I provide a sensitivity analysis regarding this assumption, and highlight the importance of heterogeneous EISs for capturing the term structure dynamics.

Setup. I consider an exchange economy in which time is continuous, denoted by $t > 0$. Uncertainty in the economy is characterized in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a standard filtration. There is a single perishable good, the numeraire. Aggregate endowment of this good follows a Geometric Brownian Motion (GBM)

$$\frac{dy_t}{y_t} = \mu dt + \sigma dW_{1,t},\ y_0 > 0\,, \tag{1}$$

where $W_1 = \{W_{1,t} \in \mathbb{R}; \mathcal{F}_t, t \geq 0\}$ is a Brownian motion on $(\Omega, \mathbb{P}, \mathcal{F})$ representing aggregate uncertainty, and parameters $\mu > 0, \sigma > 0$ are real numbers.

The economy is populated by two classes of investors, A and B. The aggregate population remains constant and normalized to one. To obtain a stationary solution in the model, I follow Gârleanu and Panageas (2015) and I consider a simple OLG framework in line with Blanchard (1985). Investors face an exogenous death risk $\varphi > 0$, and a fraction φ of new investors are born. The probability φ is the same for all investors regardless of age, preference, or wealth. Of the newly born investors, a constant fraction $\overline{x} \in (0, 1)$ is of type A, while $1 - \overline{x}$ is of B-type. Newly born investors receive a "start-up" endowment, perfectly tradable, in order to begin their operations in financial and goods markets.

Intuitively, since risk-tolerant investors operate with leverage in equilibrium, their net worth grow faster when there is a sequence of positive returns. This implies that they can end up dominating the economy (or disappearing, if the sequence of shocks is sufficiently negative). The OLG setup prevents this outcome without changing the fundamental risk-sharing properties driven by preference heterogeneity.[3]

To insure against exogenous death risk, investors can write contracts with perfectly competitive insurance companies. The possibility of insuring against death risk, together with the financial instruments specified below, implies that this economy has complete markets. The contract specifies that the investor receives a flow of resources φ, proportional to his net worth, per unit of time. For this, he agrees to pay his entire net worth to the insurance company upon his death. Investors find it optimal to sign this contract, provided they have no bequest motives (Blanchard, 1985). As I show below, this device is useful to introduce stationarity in the model. I next introduce investors' preferences and balance sheets.

Preferences and balance sheets. Investors feature recursive preferences, as in Duffie and Epstein (1992). For each investor i, his utility function $\mathcal{U}_{i,t}$ is given by

$$\mathcal{U}_{i,t} = E_t^{\mathbb{P}}\left[\int_t^{\infty} f\left(c_{i,u}, \mathcal{U}_{i,u}\right) du\right],$$

where

$$f\left(c_i, \mathcal{U}_i\right) = \frac{1}{1-1/\psi_i}\left(1-\gamma_i\right)\mathcal{U}_i\left\{c_i^{1-1/\psi_i}\left(\left(1-\gamma_i\right)\mathcal{U}_i\right)^{\frac{1/\psi_i-1}{1-\gamma_i}} - (\rho+\varphi)\right\}. \tag{2}$$

In this notation, ψ_i represents the EIS and γ_i the RA, for this investor. Also, c_i represents the flow of consumption and ρ the time preference, which is adjusted by φ (Gârleanu and Panageas, 2015). These preferences are useful because they disentangle the RA coefficient from the EIS—a crucial aspect of the model that allows me to focus on the following assumption.

Assumption 1. *In the remainder of the paper, I assume*

$$\begin{aligned} i)\ \gamma_A &< \gamma_B\,, \\ ii)\ \psi_A &> \psi_B. \end{aligned}$$

This assumption means that A-type investors are relatively more risk tolerant and are relatively more willing to substitute consumption across time. Qualitatively, this feature is implicitly assumed under time-additive constant relative risk aversion (CRRA) preferences.

Each investor continuously trades two classes of financial assets: shares on a risky claim and positions in risk-free money market account. I denote by q_t the price of the risky asset. This asset pays, each period, a unit of the endowment minus the amount of resources allocated to the "start-up" wealth of the newly born. I denote $s_{i,t}$ the number of shares a given investor holds in this asset. The price of the risky asset follows an Itô process

$$\frac{dq_t}{q_t} + \left(\frac{y_t - \varphi e_t}{q_t}\right) dt = \mu_{q,t} dt + \sigma_{q1,t} dW_{1,t}\,, \tag{3}$$

[3] There are several ways to obtain stationarity, and some papers have already used a similar OLG device I use in this paper (Drechsler et al. (2017), Dou (2017), Silva (2016), and Barro et al. (2017)). Also, Di Tella (2017) assumes that leveraged agents ("experts") face a probability of becoming unleveraged agents ("households"). Di Tella and Kurlat (2017) introduce an exogenous tax that redistributes wealth from leveraged agents to unleveraged.

where the drift $\mu_{q,t}$ and the diffusion $\sigma_{q1,t}$ are determined in equilibrium, and e_t is represents the resources for the newly born investors—which I describe below. Thus, q_t accounts for the total wealth in the economy.[4]

Let $\tilde{t}_i$ be the investor's i birth time. Since investor's optimal decisions will not depend on their age, I simplify the notation and remove explicit dependence of variables to $\tilde{t}_i$. The total net worth $n_{i,t}$ of an operating investor in period $t > \tilde{t}_i$ is given by the following accounting identity

$$n_{i,t} = q_t s_{i,t} - b_{i,t} \,, \tag{4}$$

where $b_{i,t}$ is the value of the short-term money market account held by investor i. Positions in this account receive a return of $r_t dt$—i.e., the spot real risk-free rate.

Using (3) and (4), I can write the law of motion for the net worth of an operating investor

$$\frac{dn_{i,t}}{n_{i,t}} = \left[r_t - \frac{c_{i,t}}{n_{i,t}} + \frac{s_{i,t} q_t}{n_{i,t}} \left(\mu_{q,t} - r_t \right) + \varphi \right] dt + \frac{s_{i,t} q_t}{n_{i,t}} \sigma_{q,t} dW_{1,t}, \; t > \tilde{t}_i, \tag{5}$$

and I define $\alpha_{i,t} = \frac{s_{i,t} q_t}{n_{i,t}}$ as investor's i portfolio share. Notice that investors receive φ from the insurance company that collects his wealth upon his death.

Newly born investors receive an initial level of wealth and can immediately start operating in financial and goods markets . These resources are perfectly tradable. I follow Gârleanu and Panageas (2015) and assume that any investor, of any type, born in $\tilde{t} < t$ receives an endowment process given by $y_{t,\tilde{t}} = \omega y_t G\left(t - \tilde{t}\right)$, with $\omega \in (0, 1)$ and G a deterministic function that controls the investor's life-earning profile, specified below. Thus, in period t the present value of initial earnings (i.e., initial endowment) for an investor born today in t is

$$e_t = y_t E_t^{\mathbb{Q}} \left[\int_t^{+\infty} \exp\left(- \int_t^h r_u du \right) \omega \frac{y_h}{y_t} G\left(h - t\right) dh \right]. \tag{6}$$

The expectation is computed under the equivalent martingale measure on $(\Omega, \mathcal{F}, \mathbb{Q})$, which is guaranteed to exist since markets are complete and there are no arbitrage opportunities. At an aggregate level, in a given period t, the resources associated with initial earnings account for a total of $\hat{e}_t$ (as a share of y_t), denoted by

$$\begin{aligned} \hat{e}_t &= \frac{1}{y_t} \int_{-\infty}^t \varphi \exp\left(-\varphi\left(t - u\right)\right) e_u du \\ &= E_t^{\mathbb{Q}} \left[\int_t^{+\infty} \exp\left(- \int_t^h r_u du \right) \omega \frac{y_h}{y_t} dh \right]. \end{aligned} \tag{7}$$

The last step follows by normalizing the function $\int_{-\infty}^t \varphi \exp(\varphi\left(u - t\right)) G\left(t - u\right) du = 1$, and by a simple application of Fubini's theorem. Notice that I can write $\hat{e}_t = \omega(q_t / y_t)$. Thus, the endowment claim (total wealth) is basically the replication of two assets: aggregate earnings $\hat{e}_t$ and an asset $\hat{q}_t$ that pays a dividend

[4] I show this in the appendix.

equal to $(1-\omega)\, y_t$ per unit of time. That is

$$\widehat{q}_t = E_t^{\mathbb{Q}}\left[\int_t^\infty \exp\left(-\int_t^h r_u du\right)(1-\omega)\frac{y_h}{y_t}dh\right]. \tag{8}$$

I can now write the dynamic problem of investor i, whose birth was in $\widetilde{t}_i$, as

$$\begin{gathered}\max_{\{s_i, c_i\}} \mathcal{U}_{i,t} \\ \textit{subject to} \\ (5), (6),\end{gathered}$$

where the control variables are the number of shares on the endowment claim, s_i, and the consumption flow, c_i. I next define a competitive equilibrium.

Definition 1 (Competitive equilibrium) *A competitive equilibrium is a set of adapted stochastic processes for the investor's problem $c_A, c_B, \alpha_A, \alpha_B$, and a set of prices r, q such that: (1) Given prices, policy functions solve investors' problem; (2) and the goods and asset market clears (money market clears by Walras' Law)*

$$\begin{aligned}\int_{\mathbb{A}_t} c_{i,t}di + \int_{\mathbb{B}_t} c_{i,t}di &= y_t\,, \\ \int_{\mathbb{A}_t} s_{i,t}di + \int_{\mathbb{B}_t} s_{i,t}di &= 1\,,\end{aligned}$$

where $\mathbb{A}_t$ and $\mathbb{B}_t$ are the sets of investors A and B in period t, respectively.

4 Solving for the Equilibrium

The purpose of this section is to represent the model in a recursive fashion. The equilibrium is characterized by the endogenous distribution of net worth across investors. However, the state space can be simplified by using the fact that investor's optimal choices are linear in their net worth and that investor's death risk is independent of their age. This implies investor's within a preference type undertake the same actions. Thus, I can derive the equilibrium conditions as a function of the following endogenous state variable

$$x_t = \frac{n_A}{n_A + n_B}, \tag{9}$$

where $n_{A,t} = \int_{\mathbb{A}_t} n_{i,t}di$ and $n_{B,t} = \int_{\mathbb{B}_t} n_{i,t}di$. The variable $x_t \in (0,1)$ is the relative market value investor A's net worth, and it captures aggregate conditions in the credit market. Intuitively, when x_t is low, the aggregate ability of risk-tolerant investors to supply credit decreases. As shown below in Proposition 3, this type of investor choose an equilibrium portfolio share that is greater than one (i.e., they are leveraged).

The law of motion of x follows from applying Itô's lemma to ratio (9). This is important for pinning down the dynamics of the endogenous variables in a Markov equilibrium. In what follows, I express all aggregate endogenous state variables as a function of x. That is, I seek to solve investors' control variables (their consumption-wealth ratios and portfolio shares), the price of the endowment claim (q/y) and the interest rate (r), as a function of x. The system of ordinary differential equations that characterize the equilibrium consists of investors' value function (Hamilton-Jacobi-Bellman equations), the no-arbitrage conditions for total wealth and initial wealth, together with the market clearing conditions for consumption and shares (the money market account clears by Walras' Law).

Proposition 1 (Law of motion for x). *The endogenous state variable x follows an Itô process*

$$dx_t = \mu_{x,t}dt + \sigma_{x,t}dW_{1,t} \,, \tag{10}$$

where

$$\begin{aligned} \mu_{x,t} &= x_t\left(1-x_t\right)\left(\frac{c_{B,t}}{n_{B,t}} - \frac{c_{A,t}}{n_{A,t}} + \left(\alpha_{A,t}-\alpha_{B,t}\right)\left(\mu_{q,t} - r_t - \sigma_{q1,t}^2\right)\right) + \frac{\varphi \widehat{e}_t}{pd_t}\left(\overline{x} - x_t\right), \\ \sigma_{x,t} &= x_t\left(1-x_t\right)\left(\alpha_{A,t}-\alpha_{B,t}\right)\sigma_{q1,t}, \\ x_0 &\in (0,1)\,, \end{aligned}$$

with functions $\alpha_{A,t} = \alpha_A\left(x_t\right); \alpha_{B,t} = \alpha_B\left(x_t\right); \frac{c_{A,t}}{n_{A,t}} = \frac{c_A}{n_A}\left(x_t\right); \frac{c_{B,t}}{n_{B,t}} = \frac{c_B}{n_B}\left(x_t\right); r_t = r\left(x_t\right); \mu_{q,t} = \mu_q\left(x_t\right); \sigma_{q1,t} = \sigma_{q1}\left(x_t\right); q/y = pd\left(x_t\right)$. The initial x_0 is a number in $(0,1)$. Provided $\mu_{x,t}$ and $\sigma_{x,t}$ satisfy the usual uniform Lipschitz and linear growth condition in x, then the stochastic differential equation (10) is strong Markov and has a unique solution.

Proof. See appendix.

Notice that the second term in the drift function $\mu_{x,t}$ is due to the demographic structure assumed above. This term is key for obtaining an invariant distribution of x. Informally, notice that for very small values of x, the diffusion tends to zero and the drift becomes larger and positive. Thus, the process never reaches zero. Similar logic implies an upper boundary at one.[5]

The diffusion term, $\sigma_{x,t}$, depends on the differences in investors' portfolio shares. If $\alpha_{i,t}$'s were the same for both investors, then $dW_{1,t}$ shocks would not affect x. As a result, when the economy reaches the stochastic steady state (i.e., when $\mu_{x,t} = 0$), it remains there. This implies that differences in investors' exposure to aggregate risk are critical for obtaining fluctuations in the wealth distribution in this setup.

Hamilton-Jacobi-Bellman Equation and investors' first order conditions. The investor's problem can be written recursively

$$0 = \max_{c_i, s_i} f\left(c_{i,t}, \mathcal{U}_{i,t}\right) + E^{\mathbb{P}}\left[d\mathcal{U}_{i,t},\right] \tag{11}$$

subject to his budget constraint (5) and his initial wealth (6). To solve the recursive problem, I appeal to the homotheticity properties of the value function and the constraints. This implies that the value function

[5] Technically, the second term changes the speed of the process at the boundary. See Karlin and Taylor (1981), chapter 15, for a discussion of the boundary behavior of Itô processes.

can be written in the following power form:

$$\mathcal{U}_{i,t}\left(x_t, n_{i,t}\right) = \frac{\left(\xi_{i,t}^{\frac{1}{1-\psi_i}} n_{i,t}\right)^{1-\gamma_i}}{1-\gamma_i}\,, \tag{12}$$

where the known function $\xi_i\left(x_t\right)$ captures the investor's valuation of the future investment opportunities. This function can be expressed as an Itô process,

$$\frac{d\xi_{i,t}}{\xi_{i,t}} = \mu_{\xi i,t}dt + \sigma_{\xi i,t}dW_{1,t}\,, \tag{13}$$

with adapted processes $\mu_{\xi i,t} = \mu_{\xi i}\left(x_t\right)$ and $\sigma_{\xi i,t} = \sigma_{\xi i}\left(x_t\right)$ determined in equilibrium. Using (12) and (13) in (11), the problem can be written with $\frac{c_i}{n_i}$ (i.e., the consumption-wealth ratio) and $\alpha_i = \frac{s_i q_t}{n_i}$ (the portfolio share) as control variables

$$\begin{aligned}
0 \quad = \quad & \max_{\left\{\frac{c_i}{n_i},\alpha_i\right\}} \frac{\psi_i}{\psi_i-1}\left(\left(\frac{c_i}{n_i}\right)^{1-\frac{1}{\psi_i}}\left(\xi_i\right)^{\frac{1}{\psi_i}} - \left(\rho+\varphi\right)\right) + E^{\mathbb{P}}\left[\frac{dn_i}{n_i}\right] - \frac{\gamma_i}{2}E^{\mathbb{P}}\left[\left(\frac{dn_i}{n_i}\right)^2\right] \\
& + \frac{1}{1-\psi_i}\left[\mu_{\xi i} + \frac{1}{2}\left(\frac{1-\gamma_i}{1-\psi_i}-1\right)\sigma_{\xi i,t}^2\right] + \left(\frac{1-\gamma_i}{1-\psi_i}\right)E^{\mathbb{P}}\left[\frac{d\xi_i}{\xi_i}\frac{dn_i}{n_i}\right]\,, \\
& \textit{subject to} \\
& (5), (6).
\end{aligned} \tag{14}$$

The first-order conditions (FOC) of this problem, for investor i, are given by

$$\frac{c_i}{n_i} \quad = \quad \xi_i\,, \tag{15}$$

$$\alpha_i \quad = \quad \frac{\mu_q - r}{\gamma_i\sigma_q^2} + \left(\frac{1-\gamma_i}{1-\psi_i}\right)\frac{\sigma_{\xi i}}{\gamma_i\sigma_q}. \tag{16}$$

Investors' demand for the risky asset consists of a "myopic" term, $\frac{\mu_q-r}{\gamma_i\sigma_q^2}$, and a "hedging" term, $\left(\frac{1-\gamma_i}{1-\psi_i}\right)\frac{\sigma_{\xi i}}{\gamma_i\sigma_q}$. In the representative agent economy, $\alpha = 1$ by market clearing. However, this is not the case in heterogeneous-investor economies in which different classes of investors can participate in the market for the risky asset. In the next proposition, I characterize the A-type investor's demand for the risky asset, and show that A-type investors operate with leverage in equilibrium if and only if $\gamma_A < \gamma_B$.

Proposition 2 (Leverage and Risk Sharing) *(1) A-type investor's demand for risky assets is given by*

$$\alpha_A\left(x\right) = \frac{1-\left(1-x\right)x\frac{R(x)}{\gamma_B}}{x+\left(1-x\right)\left[\frac{\gamma_A}{\gamma_B}-\frac{xR(x)}{\gamma_B}\right]}\,,$$

with

$$R\left(x\right) = \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} - \left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B}.$$

(2) Aggregate risk is concentrated in A-type investors (i.e., $\alpha_A > 1$), and thus positive aggregate endowment shocks increase x if and only if $\gamma_A < \gamma_B$.

Proof. See appendix.

The variable $R(x)$ above captures the risk-sharing mechanism. Mechanically, $R(x)$ can be written as the difference in the sensitivity of the value functions with respect to x. That is,

$$R(x) = \frac{d \log \mathcal{U}_A}{dx} - \frac{d \log \mathcal{U}_B}{dx}. \tag{17}$$

A negative (positive) R implies that a marginal increase in x improves the utility of B (A) relatively more. Notice that R would be zero if there were no motive to share aggregate risk (and $\alpha = 1$).

Discussion of assumption 1. There is a large literature documenting heterogeneity in EIS among individuals (see Guvenen (2006) for a summary). In general, the evidence in the literature shows that people who choose to be more exposed to aggregate risk (for example by holding stocks) exhibit a larger EIS. My assumption follows this line: in the model presented above, low-RA investors choose to be more exposed to aggregate risk and I assume they have a larger EIS. In my setup, heterogeneity in risk aversion is important because aggregate risk is concentrated in risk-tolerant agents, and therefore x increases after a positive endowment shock (i.e., $\sigma_x > 0$). Put differently, x would not react to macro shocks if $\gamma_A = \gamma_B$. In contrasts, Guvenen (2009), who also studies an economy with heterogeneous EISs, finds that differences in RA are not relevant in his results. This is because he studies an economy in which there is limited market participation. This last assumption immediately implies that stockholders (i.e., those who are allowed to trade the risky asset) concentrate aggregate risk.

The assumption is qualitatively in line with time-additive preferences featuring CRRA preferences. Under CRRA preferences, the EIS is set to be the inverse of the RA coefficient. Thus, the assumption of $\gamma_A < \gamma_B$ would immediately lead to $\psi_B < \psi_A$, as stated in assumption 1. This is consistent with Longstaff and Wang (2012), Wang (1996), and Hall (2017), among others. In the context of time-additive preferences, heterogeneous EISs can be rationalized as differences in an agent's willingness to substitute across goods (see, for example, Atkeson and Ogaki (1996)). One interpretation of $\psi_B < \psi_A$ is that type-A investors' expected consumption is more sensitive to fluctuations in spot interest rates, but less than one-to-one.[6] As I show below, the distinction between RA and EIS is crucial in capturing the quantitative properties of the yield curve.

5 Term Structure of Interest Rates

Equipped with the equilibrium definition and the model's solution, I can now characterize the term structure of interest rates in the economy. Since the economy features complete markets and there are no

[6] Suppose consumption follows an Itô process with constant drift μ_{ci} and diffusion σ_{ci} for investor i, then

$$\mu_{ci} = \psi_i (r - \rho) + (1 + \psi_i) \gamma_i \sigma_{ci}^2,$$

so the greater ψ_i, the more sensitive is expected consumption to movements in r. If $\psi_i < 1$, movements are less than one-to-one.

arbitrage opportunities, I can obtain a stochastic discount factor "as if" there were a representative agent (Constantinides and Duffie (1996)). The properties of the discount factor, characterized below in proposition 4, depend on the risk-sharing dynamics of the economy.

After deriving the discount factor, I value zero-coupon bonds. I start by analyzing the properties of real bonds, (i.e., assets that pay a unit of consumption in the future), and then extend to value nominal bonds (i.e., assets whose cash flow is in monetary units). In this analysis, money is solely a unit of account, and I assume the marginal investor can transform money into goods (and vice versa) without any friction whatsoever.

I next derive the real stochastic discount factor.

Proposition 3 *The state-price process $m(x_t) > 0$ satisfies*

$$\frac{dm_t}{m_t} = -r(x_t)\,dt - \kappa(x_t)\,dW_{1,t}\,,$$

with

$$\kappa(x_t) = \frac{\sigma_{q1}(x_t) - x_t\left(\frac{1-\gamma_A}{(1-\psi_A)\gamma_A}\right)\sigma_{\xi A}(x_t) - (1-x_t)\left(\frac{1-\gamma_B}{(1-\psi_B)\gamma_B}\right)\sigma_{\xi B}(x_t)}{\frac{x}{\gamma_A}+\frac{1-x}{\gamma_B}}\,, \tag{18}$$

$$r(x_t) = \mu_q + \frac{1}{pd(x_t)} - \kappa(x_t)\,\sigma_{q1} - \frac{\widehat{e}(x_t)}{pd(x_t)}\varphi\,, \tag{19}$$

where $r(x_t)$ and $\kappa(x_t)$ are adapted and bounded processes. The process for x is given by (10).

Proof. See appendix.

Then, I can define the process ζ_t as

$$\zeta_t = \exp\left(\int_0^t \kappa(x_u)\,dW_{1,u} - \frac{1}{2}\int_0^t \kappa(x_u)^2\,du\right), \tag{20}$$

which is a martingale in $\mathbb{P}$ and represents the Radon-Nikodym derivative $d\mathbb{Q} = \xi_T d\mathbb{P}$, provided regular conditions are verified. [7] With a standard application of Girsanov's theorem, I can define a Brownian motion in the equivalent martingale measure $\mathbb{Q}$.

To derive the real yield curve, I calculate the price of real zero-coupon bonds. Let $P_t^{(T)}$ represent the price of an asset that pays a unit of consumption in T periods from now (t) (i.e., a zero-coupon bond). So $P_t^{(0)} = 1$. Then

$$P_t^{(T)} = E_t^{\mathbb{P}}\left[\frac{m_{t+T}}{m_t}\right] \equiv E_t^{\mathbb{Q}}\left[\exp\left(\int_t^{t+T} r(x_u)\,du\right)\right] \equiv P(x_t, T)\,. \tag{21}$$

The real yield can be computed from prices as $y_t^{(T)} = -\frac{\log P_t^{(T)}}{T}$, while forward rates from T to $T+j$, $y_{f,t}^{(T\to T+j)}$, follow immediately by no-arbitrage. I next characterize the value of the real bond (21).

[7] In particular, Novikov's condition, $E^{\mathbb{P}}\left[\exp\left(\int_0^T \kappa_t(x_s)^2\,ds\right)\right] < \infty$, which holds since $\kappa(x)$ is a bounded function and x is Markov.

Problem 1 (Valuing real bonds). *The price of the real bond $P(x,T)$ (a T-real bond) solves the following Cauchy problem:*

$$\begin{aligned} -P_T'(x,T) + \mathcal{L}P(x,T) - r(x)P(x,T) - \kappa(x)P_x'(x,T)\sigma_x(x) &= 0\,, \\ P(x,0) &= 1,\ \forall x, \end{aligned} \tag{22}$$

where $\mathcal{L}$ is the differential operator in x.

From (22), the term premium of a T-quarter real zero-coupon bond is given by

$$\begin{aligned} E^{\mathbb{P}}\left[\frac{dP_t^{(T)}}{P_t^{(T)}}\right] - r_t dt &= \underbrace{-cov_t^{\mathbb{P}}\left(\frac{dm_t}{m_t}, \frac{dP_t^{(T)}}{P_t^{(T)}}\right)}_{\text{T-real term premium}}, \\ &= \underbrace{\frac{P_x'^{(T)}}{P^{(T)}}}_{>0}\underbrace{\kappa(x_t)\,\sigma_x(x_t)}_{>0}. \end{aligned} \tag{23}$$

The sign of the T-real term premium is characterized by the derivative $P_x'^{(T)}$, since $\kappa(x_t)\,\sigma_x(x_t) > 0\ \forall t$ by definition. Mechanically, bond prices are higher in states of nature in which the real rate is lower (i.e., there is an inverse relationship between zero-coupon bond prices and rates). In the model, r is high in states in which x is low. This implies that $P_x'^{(T)} > 0$. If those states correspond to high prices of risk, the market will compensate the marginal investor with a positive premium to hold a T-real bond. Below, I elaborate this intuition further, when I present the model solution and the term structure of interest rates. In particular, when I show numerical results for the covariance term (23).

Nominal Term Structure: Exogenous Inflation. I first consider the case in which inflation is exogenous, as in other papers in the macro-finance term structure literature (e.g., Piazzesi and Schneider (2006), Bansal and Shaliastovich (2013), among others). That is, I compute the nominal stochastic discount factor—which is used to discount future cash flows denominated in dollars—by introducing exogenous fluctuations in the purchasing power of a dollar (i.e., exogenous fluctuations in the price level).

The objective is to study the role of inflation risk, since in this case the nominal term premium is driven by the assumption that inflation and real shocks are negatively correlated: inflation occurs in high marginal utility states. In other words, this assumption implies that the purchasing power of nominal payments decreases precisely when the marginal investor require those resources the most. Therefore, the market has to compensate the marginal investor with a premium to hold such an asset. In the quantitative analysis below, I provide a decomposition of the nominal term premium in order to quantify the role of this negative correlation vis-à-vis the real component.

I introduce an exogenous price process p_t (CPI), as in Cox et al. (1985). That is,

$$
\begin{aligned}
\frac{dp_t}{p_t} &= \pi_t dt + \sigma_p \sigma(\pi_t)\, dW_{2,t},\ p_0 > 0\,, \\
d\pi_t &= \lambda_\pi (\overline{\pi} - \pi_t)\, dt + \sigma(\pi_t)\, dW_{3,t},\ \pi_0 > \pi_L, \\
& \quad \textit{with} \\
\sigma(\pi_t) &= \sigma_\pi \sqrt{\pi_t - \pi_L}\,,
\end{aligned}
\tag{24}
$$

where $W_2 = \{W_{2,t} \in \mathbb{R}; \mathcal{F}_t, t \geq 0\}$ and $W_3 = \{W_{3,t} \in \mathbb{R}; \mathcal{F}_t, t \geq 0\}$ are aggregate Brownian motions in the probability space $(\Omega, \mathbb{P}, \mathcal{F})$ representing shocks to inflation and shocks to expected inflation, respectively. The parameters $(\lambda_\pi, \overline{\pi}, \pi_L)$ are real numbers, and are associated with the persistence, mean, and the lower bound on inflation. Importantly, the exogenous process π_t is stationary (see appendix).

I assume that processes W_1 and W_3 are correlated; that is, $\langle dW_1 dW_3 \rangle_t = \phi_{13} dt$. In particular, I assume that $\phi_{13} < 0$, so shocks to *pi* and shocks to aggregate endowment are negatively correlated (Piazzesi and Schneider, 2006). This implies that a nominal asset is expected to produce lower real payments (i.e., inflation erodes the purchasing power of nominal payments) in periods of low growth, which creates persistent inflation risk. I assume that contemporaneous shocks to the CPI process are uncorrelated with W_1 and W_3. Similarly, I assume that W_2 and W_3 are uncorrelated. It is worth emphasizing that $\langle dW_2 dW_3 \rangle_t = \langle dW_1 dW_2 \rangle_t = 0$ is without loss of any generality, either from a quantitative or a qualitative perspective. This is because these shocks are i.i.d., so they have a minor role (whereas dW_3 are persistent). I assume this to focus on the role of persistent inflation risk.

Then, I can define a nominal pricing kernel, $m_t^\$ = m_t / p_t$. Using Itô's lemma

$$
\begin{aligned}
\frac{dm_t^\$}{m_t^\$} &= \frac{dm_t}{m_t} - \frac{dp_t}{p_t} + \left(\frac{dp_t}{p_t}\right)^2 - \frac{dp_t}{p_t}\frac{dm_t}{m_t}, \\
&= -i_t dt - \kappa_t dW_{1,t} - \sigma_p dW_{2,t}\,,
\end{aligned}
$$

where i_t represents the nominal interest rate

$$
i(x_t, \pi_t) = r(x_t) + \pi_t - \sigma_p^2 \sigma(\pi_t)^2\,. \tag{25}
$$

Notice that (25) is the Fisher equation, plus an "Itô adjustment", $\sigma_p^2 \sigma(\pi_t)^2$, that is quantitatively small. With these elements, I next value zero-coupon nominal bonds. Let $P_t^{\$,(T)}$ be the price of a nominal zero-coupon bond paying one dollar T periods from now. Thus

$$
P_t^{\$,(T)} = E_t^{\mathbb{P}}\left[\frac{m_{t+T}^\$}{m_t^\$}\right] \equiv E_t^{\mathbb{Q}}\left[\exp\left(\int_t^{t+T} i(x_u, \pi_u)\, du\right)\right] \equiv P^\$(x, \pi, T)\ .
$$

Problem 2 (Valuing nominal bonds: Exogenous inflation). *The price of the nominal bond $P^\$(x, \pi, T)$, a*

T-nominal bond when inflation is exogenous, solves the following Cauchy problem:

$$\begin{aligned} -P_T'^{\$}(x,\pi,T)+\mathcal{L}P^{\$}(x,\pi,T)-i(x,\pi)P^{\$}(x,\pi,T) &= \kappa(x)\left(P_x'^{\$}(x,\pi,T)\sigma_x+P_\pi'^{\$}(x,\pi,T)\sigma(\pi)\phi_{13}\right) \\ P^{\$}(x,\pi,0) &= 1,\ \forall(x,\pi)\ , \end{aligned} \tag{26}$$

where $\mathcal{L}$ is the differential operator in x and π.

Equation (26) shows that the nominal term premium can be decomposed into a real component and a nominal component. That is,

$$\begin{aligned} T-nominal\ term\ premium &= -cov_t^{\mathbb{P}}\left(\frac{dm_t^{\$}}{m_t^{\$}},\frac{dP_t^{\$,(T)}}{P_t^{\$,(T)}}\right) \\ &= \overbrace{\underbrace{\frac{P_x^{\$,(T)}}{P^{\$}}}_{>0}\underbrace{\sigma_x(x_t)\kappa(x_t)}_{>0}}^{\text{real}}+\overbrace{\underbrace{\frac{P_\pi^{\$,(T)}}{P^{\$}}}_{<0}\underbrace{\phi_{13}}_{<0}\underbrace{\sigma(\pi_t)\kappa(x_t)}_{>0}}^{\text{nominal}} \end{aligned} \tag{27}$$

Both terms in (27), the real and the nominal, are positive. The real component is positive primarily because $P_x'^{\$,(T)}>0\,\forall(x,\pi,T)$, and the intuition is the same as the one described above for the real bond. The sign of the nominal component, however, depends on the sign of the correlation between endowment shocks and inflation expectation shocks, ϕ_{13}. This is because $P_\pi'^{\$,(T)}<0\,\forall(x,\pi,T)$: An increase in inflation expectation increases the spot nominal rate (via the Fisher identity established in (25)). Thus, the price of the nominal bond price, for any finite maturity, decreases when inflation expectations increases—i.e., the derivative with respect to π is negative across the state space. But since $\phi_{13}<0$, then positive endowment, or "supply," shocks are associated with negative shocks to inflation expectations. Economically, this means that nominal payments are expected to be eroded by inflation during periods in which investors value those resources the most. So the sign of the $\phi_{13}P_\pi'^{\$,(T)}$ determines the sign of nominal component of the nominal term premium.

The Nominal Term Structure: Endogenous Inflation. Instead of extending the state space by adding an exogenous inflation process, another alternative is to derive a process for π_t via a simple monetary policy rule, conducted by a monetary authority. Thus, I consider a monetary authority that determines the inflation rate $\frac{dp_t}{p_t}$ in a way that is consistent with the marginal investor's stochastic discount factor (e.g., Gallmeyer et al. (2007)). For this, I consider a standard specification of such a rule in the form of a so-called Taylor rule

$$i_t^{MP}dt=\delta_0dt+\delta_\pi\left(\frac{dp_t}{p_t}-\overline{\pi}dt\right)\ , \tag{28}$$

where i_t^{MP} represents the monetary policy rate, δ_0 is a constant ("intercept"), and δ_π is the "Taylor loading," $\overline{\pi}$ the inflation target, and $\frac{dp_t}{p_t}=\pi_tdt$ is the instantaneous change in the CPI.[8] Since I consider a fully flexible-prices endowment economy, there is no output gap in this rule.

[8]The monetary authority implements the rule such that, in equilibrium, the stochastic process for the price level p_t is locally "smooth," i.e., $\sigma_{p,t}=0$. That is, the monetary policy is consistent with the conditional expectation of the stochastic discount factor.

The nominal interest rate i_t^{MP} has to clear the nominal bond market, and for this it must be consistent with the nominal pricing kernel. This implies

$$\begin{aligned} i_t^{MP} dt &= -E_t^{\mathbb{P}}\left[\frac{dm_t^{\$}}{m_t^{\$}}\right], \\ \delta_0 + \delta_\pi\left(\pi_t - \overline{\pi}\right) &= r\left(x_t\right) + \pi_t\,, \end{aligned} \tag{29}$$

which is the standard Fisher equation. Thus, I can solve for the endogenous $\pi\left(x_t\right)$ by solving (29). That is,

$$\pi\left(x_t\right) = \frac{\delta_0 - \delta_\pi\overline{\pi}}{\left(1-\delta_\pi\right)} + \frac{r\left(x_t\right)}{\delta_\pi - 1}. \tag{30}$$

Equation (30) shows that under $\delta_\pi = 1$, inflation expectations are not well defined (i.e., a version of the Taylor principle is violated). Then, using (30), the nominal interest rate takes the form of

$$i_t = i_t^{MP} \equiv \delta_0 + \delta_\pi \frac{\delta_0 - \delta_\pi\overline{\pi}}{\left(1-\delta_\pi\right)} + \frac{\delta_\pi}{\delta_\pi - 1} r\left(x_t\right).$$

This means that when $\delta_\pi > 1$ (which is commonly used in the literature) the loading on the real component, $\frac{\delta_\pi}{\delta_\pi - 1}$, is greater than one. In other words, the nominal interest rate magnifies fluctuations in the real risk-free rate.

With the derived $\pi_t = \pi\left(x_t\right)$, I can value nominal bonds. It is worth emphasizing that inflation is not a state variable to value nominal bonds, as opposed to (26). Instead, the sole state variable (other than time to maturity) is x_t. That is, $P_t^{\$,(T)} = P^{\$}\left(x, T\right)$. This implies that the problem of valuing nominal bonds is similar to (22).

Problem 3 (Valuing nominal bonds: Endogenous inflation) *The price of the nominal bond $P^{\$}\left(x,T\right)$, a T-nominal bond when inflation is endogenous, solves the following Cauchy problem:*

$$\begin{aligned} P_T'^{\$}\left(x,t\right) + \mathcal{L}P^{\$}\left(x,T\right) - i^{MP}\left(x\right)P^{\$}\left(x,T\right) &= \kappa\left(x\right)P_x'^{\$}\left(x,T\right)\sigma_x \\ P^{\$}\left(x,0\right) &= 1,\ \forall x\,, \end{aligned} \tag{31}$$

where $\mathcal{L}$ is the differential operator in x.

Before concluding this section and proceeding to the quantitative analysis, I provide a proposition for the representative agent benchmark. In that case, all prices and quantities can be solved in closed form. Under this benchmark, real yields are constant and exhibit zero volatility.

Proposition 4 (Infinitely lived investor). *If preferences are the same (i.e., $\gamma_A = \gamma_B$ and $\psi_A = \psi_B$) and there is no mortality risk (i.e., $\varphi \to 0$), then*
(i) the real risk-free rate is constant $r_t = \overline{r}$, with

$$\overline{r} = \rho + \frac{\mu}{\psi} - \left(1 + \frac{1}{\psi}\right)\frac{\gamma\sigma^2}{2};$$

(ii) the real term structure is flat and the volatility of yields is zero at all maturities

$$\begin{aligned} y_t^{(T)} &= r_t = \bar{r}, \ \forall (t, T) \ , \\ var\left(y_t^{(T)}\right) &= 0, \ \ \forall (t, T) \ ; \end{aligned}$$

(iii) the price-dividend ratio is constant, $pd_t = \overline{pd}$ *;*
(iv) under exogenous inflation, the nominal term structure depends on inflation expectations only. Nominal bond prices can be solved in closed form and equal to

$$P^{\$,(T)}(\pi, t) = A(t) \exp\left(B(t)\pi + C(t)\sqrt{\pi - \pi_L}\right) \ ,$$

where coefficients $A(t)$ *,* $B\ (t)$*, and* $C(t)$ *solve the system reported in the appendix; and* [9]
(v) under endogenous inflation, the nominal term structure is flat and the volatility of nominal yields is zero at all maturities.
Proof. See appendix.

6 Quantitative analysis

In this section I explore the quantitative properties of the model. To that end, I solve the model—and the corresponding partial differential equations for bond prices—numerically. I use a global solution technique based on spectral methods (Trefethen, 2000; Boyd (2001)). I start by describing the calibration procedure and then discuss the model's solution. I continue with an analysis of the real term structure, and conclude by studying the nominal term structure (with both exogenous and endogenous inflation).

Calibration. I report the calibration in Table 2, in which I divide parameters into groups: preferences, endowment and demography, and inflation. I calibrate parameters at a quarterly frequency.

Regarding preferences, there are mainly four parameters: $\gamma_A, \gamma_B, \psi_A$, and ψ_B. I set $\gamma_B = 10 > \gamma_A = 1.5$, which implies, on average, an aggregate γ of 5.1.[10] These values for risk aversion are within the range that have been used in the asset-pricing literature. Regarding the EIS, I set values for the ψ_A and ψ_B as free parameters, and explore different alternative specifications below. Intuitively, the larger the difference between ψ_A and ψ_B (ceteris paribus), the larger the increase of the spot real rate after an endogenous reduction in aggregate credit. In the baseline calibration, I use a ψ_A, ψ_B very similar to those in Gârleanu and Panageas (2015).

I calibrate the endowment parameters, the drift μ and diffusion σ, to match the mean and volatility of time-integrated U.S. consumption data. Regarding demographic parameters, I set a value of φ such that investors have an expected time operating in the financial market of 30 years, and lastly I

[9] The resulting partial differential equation for the nominal bond price is similar to the one resulting from the "double squared root process" for the interest rate. I thank Francis Longstaff for bringing this point to me.
[10] Aggregate γ is displayed in the denominator (18), which is the inverse of $\frac{x}{\gamma_A} + \frac{1-x}{\gamma_B}$.

set $\overline{x} = 0.11$ which stabilizes the share of risk-tolerant agents net worth around 0.15 (I report the invariant distribution below). Lastly, I specify a function $G(t) = G_1 e^{-g_1 t} + G_2 e^{-g_2 t}$ (i.e., a double exponential) to be consistent with the hump-shaped income pattern over the life cycle of the investor. I set $(G_1; G_2) = (30.72/4; -30.29/4)$, which implies a similar pattern to that of Gârleanu and Panageas (2015), but at a quarterly frequency.

To derive the endogenous inflation expectations, I calibrate the Taylor coefficient δ_π=1.5 as a baseline (see Taylor (1993), and many others). I set the inflation target $\overline{\pi}$=0.005, which implies a 2% annual target. For the exogenous inflation expectation process, I set parameters $\sigma_\pi, \theta_\pi, \pi_L, \overline{\pi}$ such that the mean of inflation expectations is 0.9% per quarter to match the level of the nominal yield curve, and inflation spends 99% of time in the range [-0.5%, 2%] in quarterly terms. This captures the dynamics of observed inflation. I plot the invariant distribution for π in the appendix. Lastly, I set the correlations between inflation, endowment, and inflation expectation shocks as free parameters to illustrate their role in the decomposition between nominal and real term premium. In particular, I set $\phi_{12} = \phi_{23} = 0$, and I focus on the correlation between shocks to inflation expectations and shocks to the real economy. I set $\phi_{13} = -0.5$ as a plausible lower bound on this correlation, as many previous studies have found a greater number in the data (for example, Piazzesi and Schneider (2006) find -0.2).[11]

The Economic Mechanism in the Model. Figure 1 shows the solution of the relevant objects in the model that summarize the economic mechanism. First, notice that both the real risk-free rate and the price of risk move in tandem across the state space: Real interest rates are high (bond prices are low) when the aggregate price of risk is high. This occurs when the relative market value of risk-tolerant equity is low, which implies that the total amount of credit in the economy is low.

As shown in Proposition 3, negative aggregate shocks affect risk-tolerant investor's net worth relatively more. When risk-tolerant investors lose net worth, their ability to supply credit at an aggregate level is reduced. Thus, total credit as a fraction of total equity in the economy goes down. This produces an increase in the price of credit—the real rate —because the market has to compensate agents with a lower EIS to smooth consumption over time. On the other hand, since relatively more risk-averse investors are clearing the market, the price of the risky asset goes down and the aggregate price of risk increases, as shown in the Figure.

From a risk-sharing perspective, $R(x)$ represents how changes in x affect investors' utility, as shown in (17). In Figure 1, $R(x) < 0$ across the state space, which means

$$\begin{aligned} R(x) &= \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} - \left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B} < 0 \\ &\Rightarrow \\ \frac{d\log\mathcal{U}_A}{dx} &< \frac{d\log\mathcal{U}_B}{dx} \end{aligned}$$

Thus, changes in x improve B-type investor's utility relatively more: An increase in x implies that they have to bear less aggregate risk, and since they dislike risk relatively more, their utility increases relatively more

[11] The closer ϕ_{13} is to zero, the smaller the nominal component of the nominal term premium. Even with $\phi_{13} = -0.5$, as discussed below, the nominal component is already relatively small in my model.

than that of an A-type. If there were no gains from sharing risk, R would be zero.

The Real Yield Curve. Figure 2 shows the results for the term structure up to 80 quarters. On average, the slope of the real yield curve is upward sloping: Real bonds are risky assets, since bond prices go down in states in which the price of risk increases. I report the term premium (i.e., the covariance in expression (22)) below. Interestingly, the yield curve features endogenous fluctuations across the business cycle. In particular, when there is a contraction in risk-tolerant investors' net worth (i.e., low x), the level of the real yield curve increases and its slope becomes negative. That is, x is negatively correlated with the level factor, and positively correlated with the slope of the curve.

A useful way to further understand the implications of the model is to study the interest rate dynamics. That is, $r(x_t)$, where the state variable x_t follows the law of motion in (10). Then, using Itô's lemma, the interest rate dynamics are given by

$$\begin{aligned} dr_t &= \mu_{r,t} + \sigma_{r,t} dW_{1,t}, \\ & \quad with \\ \mu_{r,t} &= r'_x \mu_{x,t} dt + \frac{1}{2} r''_{xx} \sigma^2_{x,t}, \\ \sigma_{r,t} &= r'_x \sigma_{x,t} \, . \end{aligned} \tag{32}$$

Panel (b) of Figure 2 shows the drift and diffusion associated with r. In particular, notice that the expected change of r, $\mu_{r,t}$, becomes more negative when x decreases; because real rates are mean reverting, they expected to fall. The expectation that the short-term rate will decrease in the futures is strong enough to imply the reversion of the slope in panel (a).

I next study the term premium. Long-term rates consists of two components: the expectations of short rate dynamics and the term premium. More precisely, the premium a long-term bond commands is represented in equation (23),

$$E^{\mathbb{P}} \left[\frac{dP_t^{(T)}}{P_t^{(T)}} \right] - r_t dt = -cov_t^{\mathbb{P}} \left(\frac{dm_t}{m_t}, \frac{dP_t^{(T)}}{P_t^{(T)}} \right).$$

In Figure 3 I show the model's prediction for this covariance. The left-hand panel shows the covariance across the state space, for three different maturities (4, 20, and 80 quarters). The larger the horizon, the larger the premium the bond carries. Intuitively, the longer the horizon of the bond, the more likely it will lose value in a bad state at some point of its lifetime. The right-hand panel of the figure shows the mean term premium across horizons. This panel conveys the idea that long-term bonds are riskier than that of short-term, and therefore should pay a higher return on average.

I next study the very long end of the yield curve, which may have several practical purposes (from social security to government budget projections). To that end, I first solve the real term structure that matches the short part (up to 40 quarters), but up to a horizon where the yield curve becomes almost flat.[12] Then, I compute the volatility of 10-year forward contracts, which can be easily derived from bond

[12] I show the properties of a perpetual consol bond in the appendix.

prices. Figure 4 shows the real term structure up to 800 quarters (i.e., 200 years), and the volatility of 10-year forward rates. The figure shows that the average real term structure becomes flat at nearly 700 quarters, and forward rates have substantial volatility up to 10-year contracts between 280 quarters to 320 quarters (i.e., 70 years to 80 years).

Lastly, Table 3 displays the theoretical moments and illustrates the role of EIS heterogeneity in the model. As shown in Figure 2, the model matches the slope and volatility of the real term structure. Qualitatively, the model captures the fact that the volatility of real yields is downward sloping, although the volatility of the 40-quarters yield is higher than in the data (42 basis points versus 30 basis points at a quarterly frequency, respectively). The table also illustrates the risk-sharing mechanism that drives this result. Indeed, under the baseline calibration, A-type investors consume 0.0108 of their net worth (represented by ξ_A), whereas B-type consume a higher fraction, 0.0156. This is basically due to the fact that A-type investors are operating with leverage, so they consume a smaller fraction of their net worth on average. Also, as expected, the volatility of the consumption-wealth ratio of B-type investors is larger (they are less willing to smooth consumption intertemporally). This implies that investors are sharing aggregate risk, a measure denoted by R in the Table.

When B-type investors feature CRRA preferences, i.e., $\psi_B = 1/\gamma_B$, the gains from sharing risk are lower: Fluctuations in x have a relatively similar impact on investors' utility (R is close to zero). Table 3 shows that the consumption-wealth of both agents is relatively similar and less volatile than in the baseline calibration. This implies that leveraged investors are borrowing against investors who have a similar willingness to smooth consumption intertemporally, and therefore the equilibrium spot rate does not fluctuate much. As a consequence, the volatility of yields is roughly 3 times lower than in the data, and the slope of the yield curve is significantly smaller: Long-term bonds are less risky, since interest rates (and real bond prices) are not expected to fluctuate much.

In the case of $\psi_B = \psi_A$, investors are heterogeneous along the RA dimension only. This implies that the volatility of real yields is virtually zero, and the slope of the real yield curve is almost flat (indeed, slightly downward sloping).

The Nominal Yield Curve with Exogenous Inflation. Figure 5 shows the results for the nominal yield curve when inflation follows an exogenous stochastic process. The left panel fixes x at the steady-state value and displays the nominal yield curve for different values of inflation expectations in the bivariate stationary density (shown in Figure 7). On average, the yield curve is upward sloping, because both the real component and the nominal components render nominal bonds risky assets, as shown in equation (27). I discuss the decomposition between these two sources below. The real source of risk is explained above. The nominal source of risk comes from $\phi_{13} < 0$: An exogenous sequence of positive inflation expectation shocks is associated with negative shocks to the real economy. This means that inflation is expected to erode the purchasing power of nominal payments precisely when the marginal investor values those resources the most.[13] Therefore, nominal bonds are risky.

Over the business cycle, the left panel of Figure 5 shows that when is π high, the nominal term structure is downward sloping. This is denoted by the gray line. Intuitively, when current inflation is high, nominal

[13] This is clear from Figure 7, where low x states (blue line on panel (b)) are associated with higher inflation states.

rates are expected to go down in the future (i.e., π is mean reverting). A similar logic applies when π is low, since in such states of nature the nominal interest rate is expected to increase. Thus, as shown by the blue line, the nominal curve is even more upward sloping than on average.

On the right-hand side of Figure 5, I show the nominal term structure when π is a steady state. The red line fixes the steady state of both x and π, which means it is the same as on the left-hand side. In this case, when π is fixed to the steady state, the properties of the nominal term structure are driven by x, so the intuition is very similar to that one developed for the real term structure.

To understand the role of inflation risk, driven by ϕ_{13}, I next study the decomposition of the nominal and real components of (27). This is a useful analysis, because in models in which the real term structure is flat, 100% of the nominal term premium is driven by inflation risk. Even more, in models in which the real term structure is downward sloping (such as the long-run risk models, e.g., Bansal and Yaron (2004)), inflation risk has to more than compensate for the negative real term premium to obtain an upward-sloping nominal curve consistent with the data.

Figure 6 illustrates this decomposition for an 80-quarter nominal bond.[14] On the left-hand side, the figure depicts the real and nominal components across the x dimension (i.e., fixing π at different values); on the right-hand side I shows the real and nominal component across the π dimension (i.e., fixing x at different values).

On average, the real component explains about 80% of the nominal term premia. As shown in the upper-left panel, an increase in x reduces the real component. This is because effective risk aversion decreases as x increases and risk-tolerant investors rebuild their balance sheets. This is scaled by the level of π: The greater π is (gray line), the smaller the real component. The upper-right panel shows this from a different perspective: It fixes x and shows the real component for different levels of π. The intuition for the dynamics over the state space is similar to the one above: An increase in π means a reduction in the real components, and this effect is scaled by the level of x.

The Nominal Yield Curve with Endogenous Inflation. Motivated by the previous decomposition, in which the real component drives the nominal term premium, I next study the nominal term structure with endogenous inflation expectations. As shown in equation (30), endogenous inflation expectations depend on policy parameters, δ_0 and δ_π, and also on the real interest rate $r(x)$. In particular, when $\delta_\pi > 1$, the nominal interest rate moves in the same direction as the real interest rate, by a factor $\frac{\delta_\pi}{\delta_\pi - 1} > 1$.

The difference in the magnitudes implies that the monetary authority anchors inflation expectations by adjusting the policy instrument more than one-to-one to fluctuations in the real economy (represented by $r(x)$); and this will be captured by fluctuations in nominal bond prices. In other words, the sensitivity of nominal bond prices to fluctuations in x will be higher than that of real bond prices. That is the derivative with respect to x is higher in a nominal bond than in a real bond

$$P_x^{\prime\$}(x,T) > P_x^{\prime}(x,T) \;\forall\,(x,T)\,. \tag{33}$$

Expression (33) implies that the Taylor coefficient magnifies the positive real term premium. In Figure

[14] Results in this decomposition are very similar for different maturities other than 80 quarters.

8, panel (a), I show the results for the nominal term structure under endogenous inflation. The average nominal yield curve, with $\delta_\pi = 1.5$, is upward sloping and in line with the evidence. Indeed, the slope of the nominal term structure in the figure is higher than that of Figure 2 (almost twice, as in the data), precisely because of the effect of $\frac{\delta_\pi}{\delta_\pi - 1}$ and its impact in the derivative of bond prices. Interestingly, the nominal term structure inherits the properties of the real economy, and thus exhibits endogenous fluctuations across the business cycle.

In panel (b) of Figure 8, I show the properties of endogenous inflation expectations and the corresponding nominal yield curves. The lower the Taylor coefficient δ_π (which can be interpreted as a relatively "loose" monetary rule), the greater the unconditional mean and standard deviation of inflation expectations. A lower coefficient then translates into a steeper nominal yield curve, because monetary policy reacts relatively more to changes in the real economy, which implies a more volatile nominal rate and thus a greater derivative $P_x^{'\$,(T)}(x,t)$. This can be seen on the right-hand side of panel (b) in Figure 8, where I show the normalized yield curves. (I normalize yields to 0 at maturity 0.)

7 Empirical Analysis

In this section I evaluate the empirical predictions of the model. I begin by extracting macro shocks from the data, using the fact that the aggregate endowment is i.i.d. in the model. I then introduce the realized sequence of shocks into the model and compute the time series of the endogenous state variable x.

The first exercise consists of regressing yields from the data onto the implied series of x. In particular, I consider two regressions that intend to capture the main theoretical prediction of the model: Yields are persistently negatively exposed to x (i.e., $P_x^{'}(x,T)$ is positive, and thus yields are negatively exposed to x). I regress yields onto x precisely to capture this sensitivity at different maturities. I then regress the slope of the term structure onto x. I compare the regressions results with the model's prediction for both the sensitivity of yields and slope.

The second exercise is to regress the short-term (1 quarter) nominal interest rates against the model's implied x, but controlling for several macroeconomic variables. In this analysis, I follow (Ang and Piazzesi (2003)), and investigate whether x contains information to explain fluctuations in the short-term nominal interest rate beyond other well-studied macroeconomic factors (GDP growth, inflation, and unemployment). In this exercise I evaluate the predictions in the endogenous inflation case, where the short-term nominal rate depends on x.

The third exercise is an application of the model to shed light on two salient interest-rates puzzles (Campbell et al., 2009): (1) the sudden spike of real rates in the Great Recession; (2) the secular decline of real and nominal long-term rates since the 80s. The purpose is to provide further evidence on the mechanism I propose, which relates the credit market with the term structure of interest rates. In these exercises, I use the time series implied by the model.

I conclude this section by comparing x with an alternative interpretation of the model. Previous literature (cited below) interprets risk-tolerant investors as financiers. According to this view, the relative net worth of financial firms should be indicative of credit conditions in the economy, and should be related to yields. I construct a "credit factor" that intends to capture this view and compare it with x.

Business Cycle: Preliminaries. I begin by feeding the model with macro shocks. To that end, I take advantage of the assumption that aggregate endowment is a geometric Brownian motion, which means that log growth rates are i.i.d. at an aggregate level. Then, shocks can be easily identified (under the null of the model):

$$\begin{aligned} d\log y_t - \left(\mu - \frac{1}{2}\sigma^2\right) dt &= \sigma dW_{1,t}, \\ \Delta \log y_t - \left(\mu - \frac{1}{2}\sigma^2\right) \Delta &= \sigma \left[W_{1,t+\Delta} - W_{1,t}\right]. \end{aligned} \tag{34}$$

I discretize Δ to make it equal to one quarter and use NIPA data for real personal consumption expenditures at quarterly frequency. Figure 11 displays the series of the index and the shocks. I then feed these shocks into the model, starting from the stochastic steady state in 1971:Q2, to obtain predictions for the endogenous state variables. I start in 1971:Q2 to be consistent the sample periods for which yields data are available (reported in Table 1).

Business Cycle, Credit, and Yields. I start by analyzing the model prediction for credit. Figure 12 shows results for credit over total equity in the model and credit over GDP in the data. The figure indicates the model captures the fluctuations in credit well. Motivated by this, I compute the time series for the endogenous state variable x to compare it with yields data. As described in the previous section, the model predicts that low x implies a high level of rates and lower slope. In the next figure I show that the data show a similar pattern.

Figure 13 compares fluctuations in the endogenous state variable in the model with yields data. The implied series for x shows a negative correlation of -0.35 with the first principal component of the real term structure, and -0.54 with the first principal component of the nominal term structure. As has been shown by many previous studies (e.g., Litterman and Sheinkman (1991)), the first principal component —the level of the curve—explains the vast majority of yield curve fluctuations (more than 90% in any sample). The figure also shows a positive correlation of 0.25 between the slope of the real term structure and the implied series for x. This correlation is weaker in the case of the nominal term structure (0.14).

Elasticities. I next study the sensitivity of yields, at different maturities, with respect to x. This is a useful first step to verify the key theoretical prediction of the model, which is that the derivative of bond prices with respect to x is positive. That is,

$$\frac{P'_x(x,T)}{P(x,T)} > 0.$$

More precisely, the idea is to use yields from data to capture the sensitivity of bonds to x

$$E^{\mathbb{P}}\left[\frac{\partial \log P(x,T)}{\partial x}\right] = E^{\mathbb{P}}\left[\frac{P'_x(x,T)}{P(x,T)}\right] \equiv E^{\mathbb{P}}\left[T.y_x^{\prime(T)}\right],$$

where the last equivalence follows from the relationship between yields and prices of zero-coupon bonds. To capture this relationship, I use data on real yields described in section 2, with maturities $N = (4, 8, 12, 20, 28, 40)$

quarters. I specify the following linear regression:

$$y_t^{(N)} = \alpha_1^{(N)} + \beta_1^{(N)} x_t + \varepsilon_{1,t}^{(N)}. \tag{35}$$

I use the model's implied series for x and yields from the data to estimate (35). Panel (A) of Table 4 shows estimates for real yields. Results indicate that coefficients are all negative and statistically significant, and they display the following pattern:

$$-\beta_1^{(10)} > -\beta_1^{(7)} > ... > -\beta_1^{(1)}.$$

In other words, long-term real yields are less sensitive to x (in absolute value). A similar pattern holds, but is mechanically opposite, for bond prices: longer-term bond prices are more sensitive to x. Intuitively, this indicates that bond prices are persistent (but stationary) processes. Yields inherit this property, but since they are proportional to maturity (we divide by N), the persistence of bonds is offset by N. The longer the maturity, the stronger this effect.

The model can capture this very well. Figure 9 shows the model's prediction for the sensitivity of yields with respect to x. The figure shows the unconditional derivative for yields $E^{\mathbb{P}}\left[y_x'^{(T)}\right]$. The left-hand panel shows the derivative of yields with respect to x, over the state space, for three different maturities. On the right-hand side, I show the unconditional mean across maturities. Both are in line with the estimations reported in Table 4. In other words, short-term yields are unconditionally more volatile (i.e., more sensitive to x) than long-term yields, both in the data and in the model.

Panel (B) contrasts the results for nominal yields. The coefficients are larger than those for real yields, which is consistent with the prediction in the endogenous inflation case (nominal bonds are more sensitive to x, because Taylor loading is greater than 1). Although the R^2 are higher, the coefficients are not statistically different from each other as, they were for the real yields.

Slope. I now evaluate whether the model's predictions for the slope of the term structure are consistent with the data. Figure 10 shows that the model predicts an average positive slope, but with a nonlinear relationship against x. The intuition comes from the mechanism elaborated on above: When x is low and real rates are high, rates are expected to fall in the future; they are mean reverting. This effect is strong enough to imply that during low-x states, long-term rates are lower than short-term short. When x is at its mean, the effect of x on the slope is close to zero (i.e., the derivative of the slope against x, at the steady state, is close to zero).

To evaluate this prediction, I compute the slope of real yields at different horizons in the data—that is, the difference in yields (i.e., the slope) as

$$slope\,(N) = y_t^{(N)} - y_t^{(4)}\text{ , for } N = (8, 12, 20, 28, 40)\,.$$

To capture the nonlinear aspect of the relationship predicted by the model, I specify the following regression

$$y_t^{(N)} - y_t^{(4)} = \alpha_3^{(N)} + \beta_3^{(N)} x_t + \beta_4^{(N)} x_t^2 + \varepsilon_t^{(N)}, \tag{36}$$

where the left-hand side represents the slope at different horizons and the quadratic term intends to capture the nonlinearity predicted by the model, and is reported in Figure 10, panel (a). In Figure 10, panels (b) and (c), I fit a kernel regression that indicates that the quadratic specification in (36) is enough to capture the nonlinearities in the data.

In Table 5, I report the estimates of (36). The coefficient associated with x is positive, but the coefficient associated with x^2 is negative and larger (in absolute value). This implies that changes in the model's endogenous state variable produce significant nonlinear changes in the slope of the real term structure. A marginal deviation of x from its mean, however, does not create a significant change in the slope. This is what the row "Net effect" reports: It evaluates whether the derivative of (36) is different from zero on average. This is consistent with the model prediction, indicating that a marginal change in x, starting from the steady state, is very small. But when x is small, an increase in x produces an increase in the slope. When x is large, a decrease in x produces an increase in the slope.

Model's x as a Macro Factor. In this subsection I evaluate the key theoretical prediction of the endogenous inflation case: I study how the short-term nominal rate changes with the endogenous state variable x. For this, I follow Ang and Piazzesi (2003) and regress the short-term nominal rate against several macro factors, in which I include x (the endogenous state variable implied by the model). The macro factors I include have been widely documented in the macro-finance term structure literature (proxies for inflation, GDP growth, and unemployment). Since Ang and Piazzesi (2003), many papers have incorporated macroeconomic variables into affine term structure models to provide an interpretation of the previous latent factor models (e.g., Litterman and Sheinkman (1991)).

Table 6 shows the correlations between short-horizon nominal yields, x, x^2, and x^3. The purpose of incorporating x^2 and x^3 in the analysis is to capture the nonlinear dynamics implied by the model. As can be seen in the table, the yields' dynamics are negatively correlated with x. This negative correlation was implicitly described in Figure 13, where I showed only the first principal component of nominal yields. Notice higher order terms are also relevant.

I then regress the one-quarter nominal rate $y_t^{\$,(1)}$ onto different macro factors f_t. The regression is specified as in Ang and Piazzesi (2003):

$$y_t^{\$,(1)} = \alpha_4 + \beta_4' f_t + v_t, \tag{37}$$

where f_t is a vector of macroeconomic factors and v_t is a shock that captures orthogonal information to macro variables (e.g., policy shocks). The factors I consider, in addition to x, are: an inflation factor, a real activity factor, the CPI core, and the unemployment gap.[15] I construct the inflation and real activity macro factors in the same way as Ang and Piazzesi (2003). This consists of computing the first principal component of various inflation and real activity indexes. The CPI core and unemployment gap are representative of the "policy factors" typically used by the Monetary Authority when considering adjusting the short-term rate (Bauer and Rudebusch, 2017), so they are also useful controls for x.

Table 7 report regressions' results. The first two columns are the specifications in which x is not in-

[15] The unemployment gap is the difference between actual unemployment and the natural rate of unemployment reported by the Congressional Budget Office.

cluded. This is a useful benchmark to compare with. Notice that in column (1) and (2), the only significant component is the one associated with inflation. This is consistent with Ang and Piazzesi (2003), who report that real activity is sensitive to the sample period considered. Also, column (1) indicates that the CPI core delivers a higher goodness of fit than that of column (2); R^2 is 0.54 with the CPI core and 0.18 with the inflation factor.

Column (3) shows the result of regressing the one-quarter nominal yield $y_t^{\$,(1)}$ onto x. As expected, based on the correlation structure reported in Table 6, the coefficient is negative and significant. The R^2 is almost the same as the regression including inflation and the real activity factor (0.17 versus 0.18, respectively). Indeed, as shown in column (5), when x is included in the regression of $y_t^{\$,(1)}$ against the inflation and real activity factors, the goodness of fit is more than twice (0.40 versus 0.18, respectively). Importantly, x remains negative and statistically significant. Also, notice that in column (5), the coefficients for inflation and real activity are 1.50 and 0.55, very close to those typically used in calibrations of the Taylor rule since Taylor (1993).[16] Column (4) shows similar results but with unemployment gap and CPI core: x remains negative and statistically significant and improves the goodness of fit (although not as much as column (4) against (2)).

In column (6) I evaluate the effect of x, x^2 and x^3. The result indicates that x and x^2 are significant, although the goodness of fit of does not increase much (it increases only 0.01). Then, in columns (7) and (8), I report the same specification as (4) and (5), but include the higher-order terms x^2 and x^3. In both (7) and (8), the introduction of x, x^2, and x^3 increasea the goodness of fit vis-à-vis (1) and (2). Importantly, x remains negative and statistically significant.

These results are in line with the theory predicted above: They imply that when short-term nominal interest rate is high in the data, the market value of leveraged risk-tolerant investors is low in the model. Even more, these results indicates that x contains information that is beyond the standard macroeconomic factors commonly studied in the literature.

Puzzle I: Sudden spike in real rates in the Great Recession. Early in Fall 2008, real rates (measured by TIPS) showed a sudden spike, and the real term structure was reversed (i.e., the short-term rate was above the long-term rates). As noted by Campbell et al. (2009), there were several institutional and liquidity influences on TIPS yields during this episode. These may have distorted, at least partially, their prices.

However, from a macroeconomic perspective, using the standard Fisher equation logic, it was evident that real rates, on impact, increased. More precisely, on December 15, 2008, the Federal Reserve set the short-term interest rate at 0%-0.25%. Also, according to the Survey of Professional Forecasters (SPF), during the first and second quarters of 2009 the one-quarter-ahead median inflation expectation was -9.5% and -2.4%, respectively (in annual terms). Through the lens of the Fisher equation, this implies a very large spot real rate. For example, in 2009:Q1,

$$r_t = \underbrace{i_t}_{=0} - \underbrace{E^{\mathbb{P}}\left[\frac{dp}{p}\right]}_{-9.5\%} \tag{38}$$

[16]This does not indicate the coefficients are identified (Backus, Chernov and Zin, 2016).

Thus, even though certain distortions may have contributed to the sudden spike in TIPS, the Fisher equation's logic also indicates that real rates actually increased.

In Figure 14, I show the model's time series predictions using the real macro shocks reported in Figure 11. The left-hand side shows the business cycle fluctuations of the 10-year real rate and the 1-year real rate. As it is evident from the plot, on average the real term structure is upward sloping (black line is above red line). The model predicts that during the Great Recession, the level of real rates increased pari passu with the drastic decrease in credit—a reduction in x which implies the aggregate willingness to substitute consumption intertemporally. Even more, it predicts an inversion of the real yield curve (red line crosses the black line). Qualitatively, this is consistent with the evidence.

During 2009, the monetary authority started to intervene in a variety of markets, and its balance sheet was multiplied by five. These interventions are not captured in the model, but several studies have argued they have affected the behavior of yields (e.g., Krishnamurthy and Vissing-Jorgensen (2011)). In the right-hand panel, I show the result of subtracting the one-year real rate produced by the model from the nominal real rate in the data. This is a proxy of inflation expectations. As shown in the figure, during the crisis the model predicts an expected deflation in line with the SPF. However, the model predict a more persistent dynamics: it takes longer for the credit market to be rebuilt.

An intuitive interpretation of the Fed's interventions during the Great Recession is that they introduced willingness to substitute consumption intertemporally into the markets. At the height of the financial crash (2008:Q4-2009:Q2), the marginal investor required a large compensation to postpone his current consumption into the future. Thus, the market would have to compensate him by providing a higher incentive (i.e., a high real rate) to perform such delay in his consumption. Since the nominal rate was set to zero, the adjusting economic force was deflation expectation (as shown in the SPF and predicted by the model). Thus, when the Fed started to intervene, those policies prevented the scenario predicted by the model, by "introducing" willingness to smooth consumption, thus reducing real interest rates—even though the credit market remained impaired.

Puzzle II: Secular decline in real and nominal long-term rates. Several papers have documented the fact that long-term nominal and real rates have been declining in the last 30 years (Caballero, Farhi and Gourinchas (2008), Bernanke, Bertaut, Demarco and Kamin (2011), Hall (2017), among others). This period also witnessed a significant increase in the size of the credit market. For example, Philippon (2015) shows that the amount of assets intermediated in the financial sector rose from approximately 2.5% of GDP in 1980 to 4% of GDP in 2008.

The theoretical mechanism in the model predicts that an increase in the amount of credit is associated with a reduction in the price of credit—i.e., the spot real rate. Put differently, a credit expansion produces an increase in aggregate EIS, and this implies that the market has to compensate the marginal investor with a lower interest rate to incentivize him to smooth consumption over time. Due to the single-factor structure of the model and the endogenous persistence in the credit market fluctuations, this reduction in the level of rates translates into a decrease in long-term real rates.

To capture these dynamics, I study a transitional dynamics exercise (e.g., King and Rebelo (1993)) by starting the economy 2 standard deviations below the stochastic steady state of the endogenous state variable x. Then, introduce the same macro shocks reported in the previous subsection and shown in Figure 11 and I compute long-term real rates in the model at each period of time. I pin down nominal rates with the same Taylor rule reported in the endogenous inflation term structure. That is, I keep the same calibration already shown above.

Figure 15 reports the results and compares them with the evidence for long-term rates. Panel (a) shows the model's prediction for credit/total equity. In particular, notice that the figure shows that the amount of credit as a fraction of total wealth in the model is approximately multiplied by two. In the same period, the data on domestic credit to private sector over GDP went from 92.4% to 188.0%[17], which indicates that the increase in credit predicted by the model is on the order of magnitude of that in the data. The red bars in panels (b) and (c) show the average dynamics of the 10-year real rate in the model and in the data. Nominal rates display a similar pattern, because they are pinned down by the same Taylor rule (with $\delta_\pi > 1$) as shown in (30). That is, the monetary authority anchors inflation expectations by moving the nominal rate in tandem with the real rate. Thus, inflation expectations are also trending downwards—which is consistent with the evidence reported in Chernov and Mueller (2012).

An Alternative Interpretation for x. Prior studies in the heterogeneous-agents literature that interpret risk-tolerant investors as financiers (see Silva (2016), Drechsler et al. (2017), Longstaff and Wang (2012), Santos and Veronesi (2016), among others). According to this interpretation, the equity of financial firms should be important to capture credit conditions in the economy; in theory, therefore, it should be useful to understand the behavior of yields. In this line, I compute the market value of the financial sector equity [18] over the total market value of equity in CRSP, and I define this as cf (credit factor):

$$cf_t = \frac{\text{market value of financial sector equity}}{\text{market value of total equity}}.$$

Under this alternative interpretation, a higher cf_t implies that a larger quantity of credit is being supplied, which translates to lower real and nominal rates. In Figure 16, I compare cf against other related measures. Panels (a) and (b) compare against the proposed measure by He, Kelly and Manela (2016), in levels and in shocks. Panel (c) constructs cf_t using the market value of equity in financial firms over the total market value of equity reported by the Flow of Funds. Panel (d) compares, at an annual frequency, with the flow of intermediated assets in the financial sector in Philippon (2015).

To understand how sensible this proposed factor is, I compare cf with the endogenous state variable in the model. For this, I proceed as before and feed the model with the macro shocks reported in Figure 11. Figure 17 compares the fluctuations in cf with the implied series for the endogenous state variable in the model, x. As shown in the figure, x and cf exhibit a high correlation (0.8). Put differently, cf in the data is high in periods in which risk-tolerant investors' balance sheets are relatively well capitalized in the model.

Thus, in this interpretation, cf could be used in term-structure empirical analysis to further understand yields' properties (with some guidance from the theory elaborated on this paper).

[17] Source: World Development Indicators http://databank.worldbank.org/wdi

[18] I consider SIC codes 60-64, which include a broad range of financial institutions.

8 Conclusion

In this paper, I propose a model where the credit market is a key macroeconomic fundamental for understanding the salient properties of the U.S. real and nominal term structure. In this, I depart from the representative agent framework and propose a general equilibrium term structure model with heterogeneous investors in which the amount of credit in the economy is key in characterizing the equilibrium.

I find that differences in investors' willingness to substitute consumption across time is critical to match the salient properties of both the nominal and real term structure. Endogenous contractions in the amount of credit lead to increases in the real interest rate and the aggregate price of risk, to incentivize investors with high risk aversion and low willingness to substitute consumption to clear the markets. Thus, real bonds are risky and they are negatively exposed to the endogenous risk created by the credit market. This implies that the marginal investor must be compensated with a premium to hold real bonds. At an aggregate level, this mechanism generates dynamics for the real rate and aggregate price of risk that can be interpreted as a representative agent with time-varying, and negatively correlated, risk aversion and elasticity of intertemporal substitution.

I provide a decomposition of the nominal term premium, between the endogenous source of risk created by the credit market and exogenous inflation shocks. I find that, consistent with recent studies, the model's real term premium explains a significant portion of the nominal term premium. Motivated by this, I derive a nominal term structure by introducing a Taylor rule. I show that when the monetary authority adjusts the nominal rate more than one-to-one to deviations of inflation from its target, this makes nominal bonds more sensitive to real risks. Thus, the nominal term structure is steeper than the real term structure for any correlation between inflation and real shocks. Put differently, the economy exhibits a significant nominal term premium, even when inflation shocks play no role.

To validate the model's key theoretical prediction, I introduce macro shocks to the model and obtain the series of the endogenous state variable. I find that fluctuations in credit in the model capture well the fluctuations in credit in the data. I use the implied series for the endogenous state variable and data for yields to evaluate the model's main theoretical predictions: the relationship of yields and slope of the term structure with respect to the endogenous state variable. I find that the data validate the model's predictions. In addition, I find that the implied series of the model's endogenous state variable contain information to explain short-term nominal interest rate variability that extends beyond well-studied macro variables (GDP, inflation, and unemployment).

I then use the model to study two interest rate puzzles: the secular decline in long term real and nominal bonds since the 1980s; and the sudden spike in real rates during the Great Recession. I show that these puzzles can be rationalized by the connection between the credit market and yields. In particular, the sudden spike in real rates during the Great Recession can be attributed to an endogenous collapse in the aggregate of credit (i.e., a drastic reduction in the aggregate elasticity of intertemporal substitution). The secular decline in real rates can be attributed to the contemporaneous increase in the amount of credit during since the 1980s. Using the Taylor rule, nominal yields inherit the properties of real yields, as discussed in Section 5. Thus, the model implies—also consistent with the evidence—a decline in inflation expectations.

This work provides several avenues for future research. For example, it provides a framework to study how unconventional monetary policies generated a reduction in real rates together with an increase in inflation expectations after Spring 2009. In the model's prediction, in which policy interventions are not incorporated, the spike in real yields would have been more persistent (the credit market takes time to rebuild). Also, incorporating the credit factor into the empirical macro-finance term structure model can improve our understanding of how monetary policy affect long-term rates through the credit channel. Lastly, the mechanism that generates time variation in the aggregate risk aversion and elasticity of intertemporal substitution can be introduced, in a reduced form, into larger scale models.

TABLE 1. Evidence

		Maturity (quarters)							
Panel A. Full Sample		4	8	12	20	28	40	80	diff(40-4)
Mean	Nominal	0.0133	0.0139	0.0144	0.0152	0.0158	0.0165	0.0174	0.0033
	TIPS	0.0043	0.0044	0.0046	0.0051	0.0055	0.0058		0.0015
St. Dev	Nominal	0.0090	0.0088	0.0085	0.0080	0.0076	0.0072	0.0068	-0.0018
	TIPS	0.0050	0.0046	0.0043	0.0038	0.0034	0.0030		-0.0020
Panel B. Short Sample I									
Mean	Nominal	0.0037	0.0042	0.0048	0.0061	0.0072	0.0083	0.0099	0.0047
	TIPS	0.0002	0.0003	0.0005	0.0013	0.0021	0.0029		0.0027
St. Dev	Nominal	0.0043	0.0040	0.0037	0.0032	0.0030	0.0028	0.0025	-0.0018
	TIPS	0.0041	0.0036	0.0033	0.0029	0.0027	0.0024		-0.0017
Panel C. Short Sample II									
Mean	Nominal	0.0161	0.0167	0.0171	0.0177	0.0182	0.0187	0.0194	0.0026
	TIPS	0.0053	0.0055	0.0057	0.0062	0.0065	0.0068		0.0015
St. Dev	Nominal	0.0075	0.0072	0.0069	0.0066	0.0064	0.0061	0.0058	-0.0013
	TIPS	0.0045	0.0039	0.0035	0.0030	0.0027	0.0024		-0.0021

NOTES: Full sample is 1971:Q3-2016:Q4. Short sample I is 2003:Q1-2016:Q4. Short sample II is 1971:Q3-2008:Q2. Numbers are in decimals, at quarterly frequency. Source: Chernov and Mueller (2012), Gürkaynak et al. (2007), and Gürkaynak et al. (2010).

TABLE 2. Baseline Calibration

PARAMETERS (QUARTERLY)

1. *Preferences*		**Value**	**Description**
	γ_A	1.5	risk aversion investor A
	γ_B	10	risk aversion investor B
	ψ_A	0.7	EIS investor A
	ψ_B	0.02	EIS investor B
	ρ	0.001/4	time preference
2. *Endowment and demography*			
	μ	0.0055	drift growth
	σ	0.019	diffusion growth
	φ	0.008	birth/death rate
	$\overline{x}$	0.11	fraction of new investors A
3. *Inflation*			
	δ_π	1.5	Taylor coefficient
	λ_π	0.08	persistence inflation expec.
	σ_π	0.012	diffusion inflation expec.
	π_L	-0.01	inflation expec. lower bound
	$\overline{\pi}$	0.009	mean inflation expec.
	ϕ_{12}	0	$\text{cov}(dW_1, dW_2)$
	ϕ_{13}	-0.5	$\text{cov}(dW_1, dW_3)$
	ϕ_{23}	0	$\text{cov}(dW_2, dW_3)$

NOTES: I describe the calibration in the main text.

TABLE 3. Theoretical Moments in the Model

		$\psi_B = baseline$		$\psi_B = 1/\gamma_B$		$\psi_B = \psi_A$	
Variable	Description	Mean	St.dev	Mean	St.dev	Mean	St.dev
Model							
ξ_A	consumption/wealth investor A	0.0108	0.0006	0.0103	0.0002	0.0101	0.0001
ξ_B	consumption/wealth investor B	0.0156	0.0022	0.0131	0.0005	0.0091	0.0001
R	risk sharing	-2.4514	1.2677	-0.0932	0.0277	-1.6006	0.1924
$\mu_q - r$	expected excess return	0.0146	0.009	0.0058	0.0019	0.001	0.002
σ_q	vol. returns	0.1311	0.0503	0.0572	0.0101	0.007	0.002
r	real risk free rate	0.0043	0.0046	0.0047	0.0016	0.0044	0.0004
$y^{(4)}$	real yield 4 quarter	0.0044	0.0046	0.0047	0.0016	0.0044	0.0004
$y^{(40)}$	real yield 40 quarter	0.0058	0.0042	0.0050	0.0015	0.0043	0.0004
$y_t^{(80)}$	real yield 80 quarter	0.0066	0.0037	0.0052	0.0014	0.0042	0.0003
Data							
$y^{(4)}$		0.0043	0.0050				
$y^{(40)}$		0.0058	0.0030				

NOTES: This table reports theoretical moments from the model and in yield's data. Numbers are in decimal, at a quarterly frequency. The first column, $\psi_B = baseline$, corresponds to the parametrization in Table 2. The second column, $\psi_B = 1/\gamma_B$, corresponds to the case in which B-type investors have CRRA preferences (i.e., $\psi_B = 1/\gamma_B = 0.1$). The third column, $\psi_B = \psi_A$, corresponds to the case in which both types of investors have the same EIS. Data for real yields are as in Table 1. Risk sharing R is as in equation (17), $R(x) = \left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A} - \left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B}$.

TABLE 4. Regression: Elasticities

A. Estimates of the price elasticity,real : $y_t^{(N)} = \alpha_1^{(N)} + \beta_1^{(N)} x_t + \varepsilon_{1,t}^{(N)}$						
	$\beta_1^{(4)}$	$\beta_1^{(8)}$	$\beta_1^{(12)}$	$\beta_1^{(20)}$	$\beta_1^{(28)}$	$\beta_1^{(40)}$
OLS estimates	-0.042***	-0.041***	-0.039***	-0.037***	-0.035***	-0.033***
Conf. Int.	[-0.076;-0.007]	[-0.070;-0.011]	[-0.066;-0.014]	[-0.059; -0.015]	[-0.054; -0.016]	[-0.049; -0.016]
R^2	0.054	0.068	0.079	0.096	0.107	0.119
B. Estimates of the price elasticity, nominal: $y_t^{\$,(N)} = \alpha_2^{(N)} + \beta_2^{(N)} x_t + \varepsilon_{2,t}^{(N)}$						
	$\beta_2^{(4)}$	$\beta_2^{(8)}$	$\beta_2^{(12)}$	$\beta_2^{(20)}$	$\beta_2^{(28)}$	$\beta_2^{(40)}$
OLS estimates	-0.125**	-0133***	-0.136***	-0.138***	-0.136***	-0.134***
Conf. Int.	[-0.174;-0.075]	[-0.177;-0.086]	[-0.178;-0.093]	[-0.176;-0.099]	[-0.172;-0.101]	[-0.166;-0.101]
R^2	0.174	0.212	0.239	0.272	0.292	0.310

NOTES: Significance at 1%, 5%, and 10% is indicated with ***, ** and *. Hubert-White standard errors. Sample period is 1971:Q3-2008:Q2 (i.e., Short sample II in Table 1), and the source of the data for yields is Chernov and Mueller (2012), Gürkaynak et al. (2007), and Gürkaynak et al. (2010). x is the implied endogenous state variable after feeding the model with the shocks described in Section 7.

TABLE 5. Regression: Term Structure Slope

$y_t^{(N)} - y_t^{(4)} = \alpha_3^{(N)} + \beta_3^{(N)} x_t + \beta_4^{(N)} x_t^2 + \varepsilon_t^{(N)}$					
	$y^{(8)} - y^{(4)}$	$y^{(12)} - y^{(4)}$	$y^{(20)} - y^{(4)}$	$y^{(28)} - y^{(4)}$	$y^{(40)} - y^{(4)}$
Constant	-0.001***	-0.010***	-0.016***	-0.020***	-0.024***
$\beta_3^{(N)}$	0.071***	0.123***	0.19***	0.239***	0.290***
$\beta_4^{(N)}$	-0.197***	-0.336***	-0.52***	-0.654***	-0.788***
Net effect	[-0.005, 0.003]	[-0.009; .0.006]	[-0.013;0.01]	[-0.016;0.014]	[-0.016;0.017]
R^2	0.050	0.051	0.054	0.059	0.066

NOTES: Significance at 1%, 5%, and 10% is indicated by ***, ** and *. Hubert-White robust standard errors. Net effect is the confidence interval for the marginal effect $\frac{d\left(y_t^{(N)} - y_t^{(4)}\right)}{dx} = \beta_3^{(N)} + 2\beta_4^{(N)} x_t$. Sample period is 1971:Q3-2008:Q2 (i.e., Short sample II in Table 1). Source of data is Chernov and Mueller (2012), Gürkaynak et al. (2007), and Gürkaynak et al. (2010). x is the implied endogenous state variable after feeding the model with the shocks described in Section 7.

TABLE 6. Correlation x and Nominal Yields

	$y^{\$,(1)}$	$y^{\$,(4)}$	$y^{\$,(20)}$	x	x^2
$y^{\$,(4)}$	0.985***				
$y^{\$,(20)}$	0.925***	0.954***			
x	-0.413***	-0.441***	-0.564***		
x^2	-0.105***	-0.116	-0.199**	0.571***	
x^3	-0.333***	-0.369***	-0.474***	0.842***	0.721***

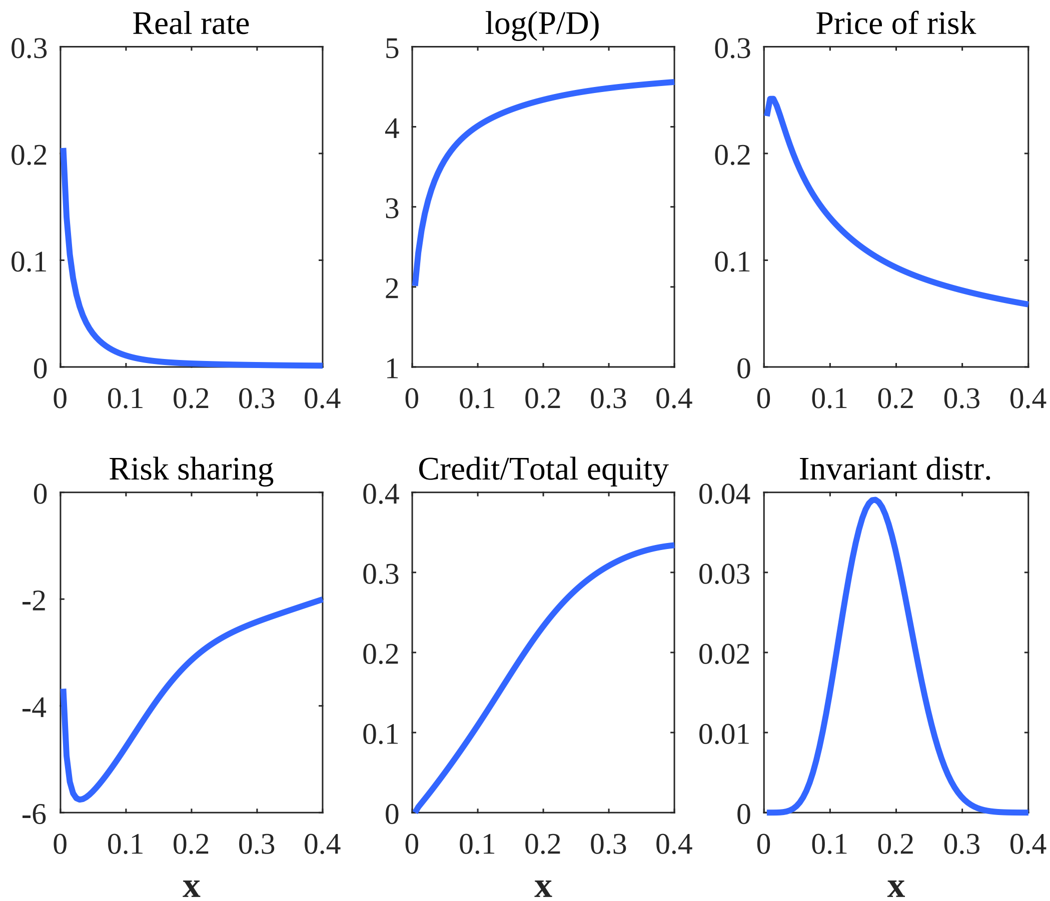
NOTES: Significance at 1%, 5%, and 10% is indicated by ***, ** and *. Sample period is 1971:Q3-2008:Q2 (i.e., Short sample II in Table 1). One quarter (3 months) nominal yield $y_t^{\$,(1)}$ is from Fama CRSP Treasury Bill files. Four-quarter and ($y_t^{\$,(4)}$) and 20-quarter ($y_t^{\$,(20)}$) Fama CRSP zero-coupon files . x is the endogenous state variable in the model, after feeding the shocks reported in Figure 11. .

TABLE 7. Short-Term Nominal Rate Regressions, $y_t^{\$,(1)} = \alpha_0 + \alpha_1' f_t + v_t$

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Constant	5.58***	5.55***	6.81***	5.52***	6.71***	5.78**	5.45***	5.94***
CPI core	2.22***			1.98***			1.94***	
Unemp.gap	-0.18			-1.02***			-1.42***	
Inflation		1.43***			1.50***			1.49***
Real activity		0.02			0.55***			0.86***
x			–2.04***	-1.67***	-2.50***	-2.11***	-2.31***	-2.53***
x^2						1.49**	1.90***	2.31***
x^3						-0.65	-0.62	-1.23***
adj-R^2	0.54	0.18	0.17	0.60	0.40	0.18	0.63	0.46

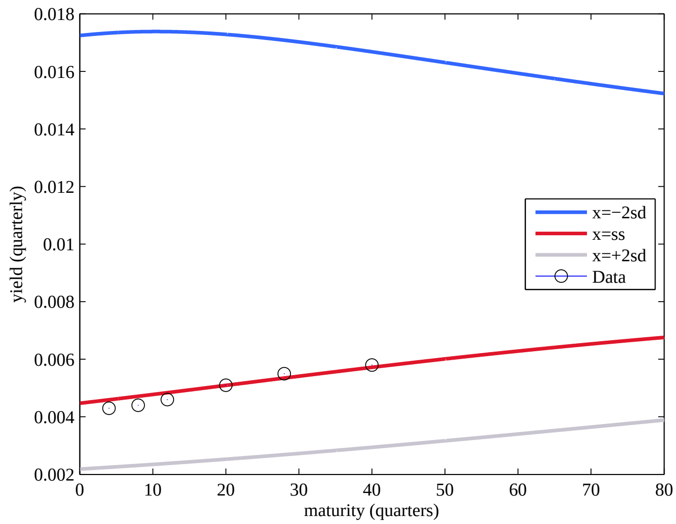
NOTES: Sample period is 1971:Q3-2008:Q2 (i.e., Short Sample II in table 1). Significance at 1%, 5%, and 10% is indicated by ***, ** and *. $y_t^{\$,(1)}$ is the 1-quarter nominal interest rate from Fama CRSP Treasury Bill files. f_t is a vector of different macroeconomic variables considered in each of the table's columns. Inflation and real activity are constructed as in Ang and Piazzesi (2003). Inflation is the first principal component of the CPI, PPI, and spot commodity prices. Real activity is the first principal component of the growth rate of employment and growth rate of industrial production. The unemployment gap is the difference between actual unemployment and the natural rate of unemployment from the Congressional Budget Office, as considered by Bauer and Rudebusch (2017). cf is described at the end of Section 7.

FIGURE 1. Model Solution

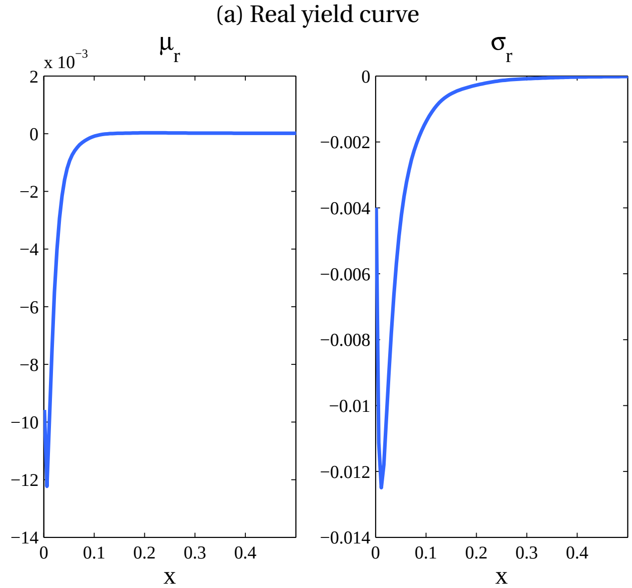


NOTES: This figure shows the model solution, with the calibrated parameters from Table 2 (quarterly frequency).

FIGURE 2. Real Yield Curve



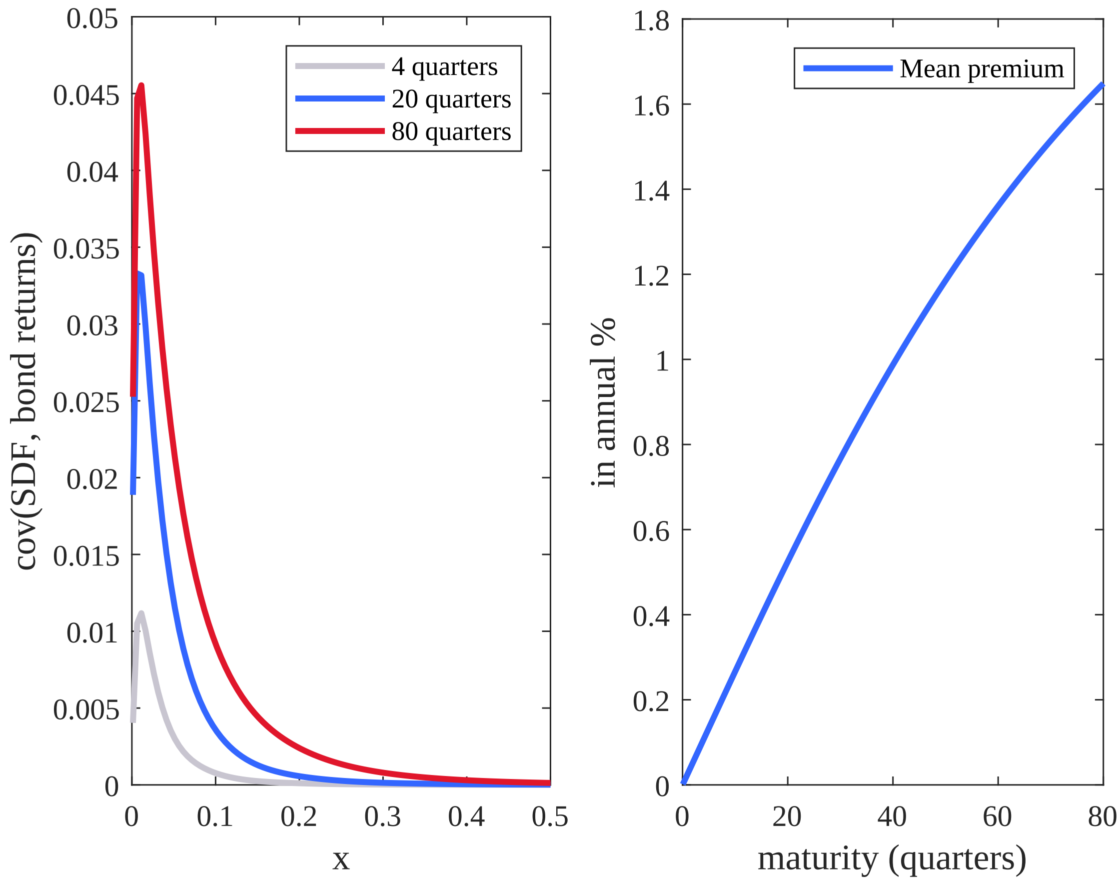
(a) Real yield curve



(b) $\mu_{r,t}$ and $\sigma_{r,t}$

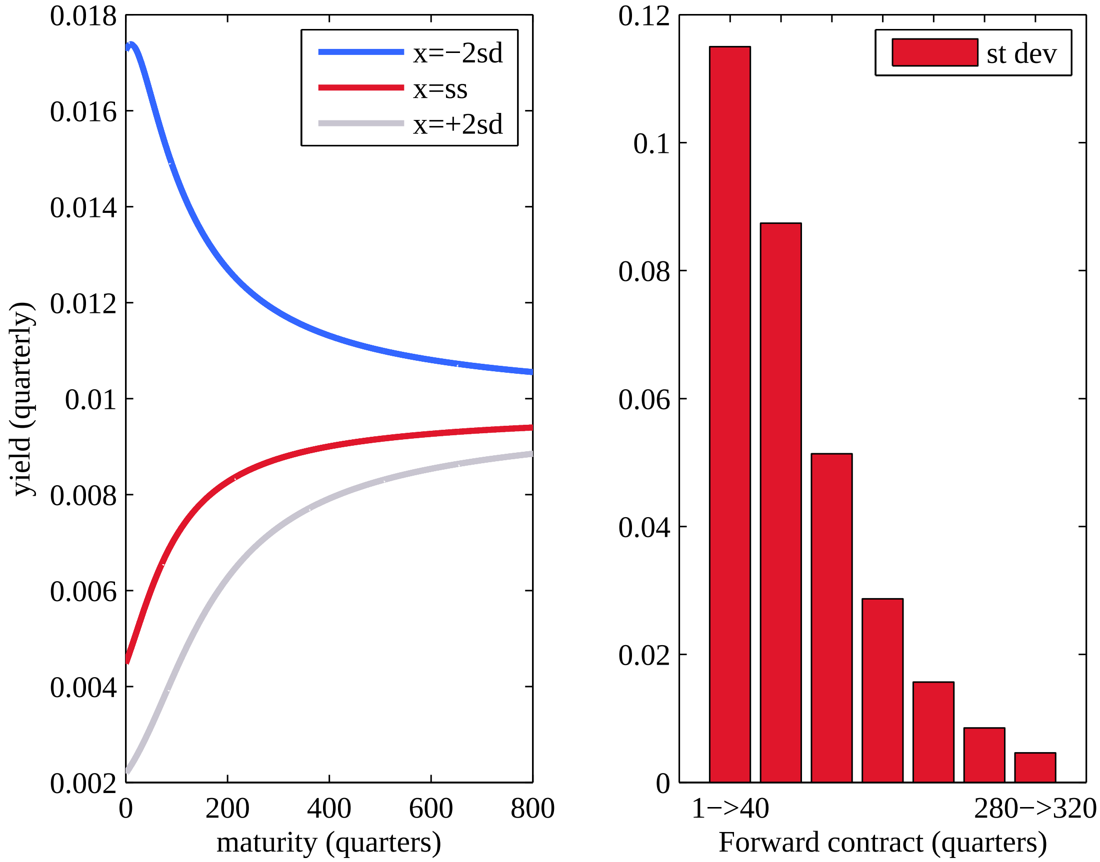
NOTES: Panel (a) displays the real yield curve in the model for three different levels of x. The blue (gray) line represents the real yield curve when x is 2 standard deviations below (above) its mean. The red line represents the real yield curve when x is at its unconditional mean. Panel (b) displays the expected change in the short-term rate, μ_r, and the diffusion for the short-term rate, σ_r, reported in (32). Data is from Table 1, full sample.

FIGURE 3. Term Premia in the Model: $cov_t^{\mathbb{P}}\left(\frac{dm_t}{m_t}, \frac{dP_t^{(T)}}{P_t^{(T)}}\right)$



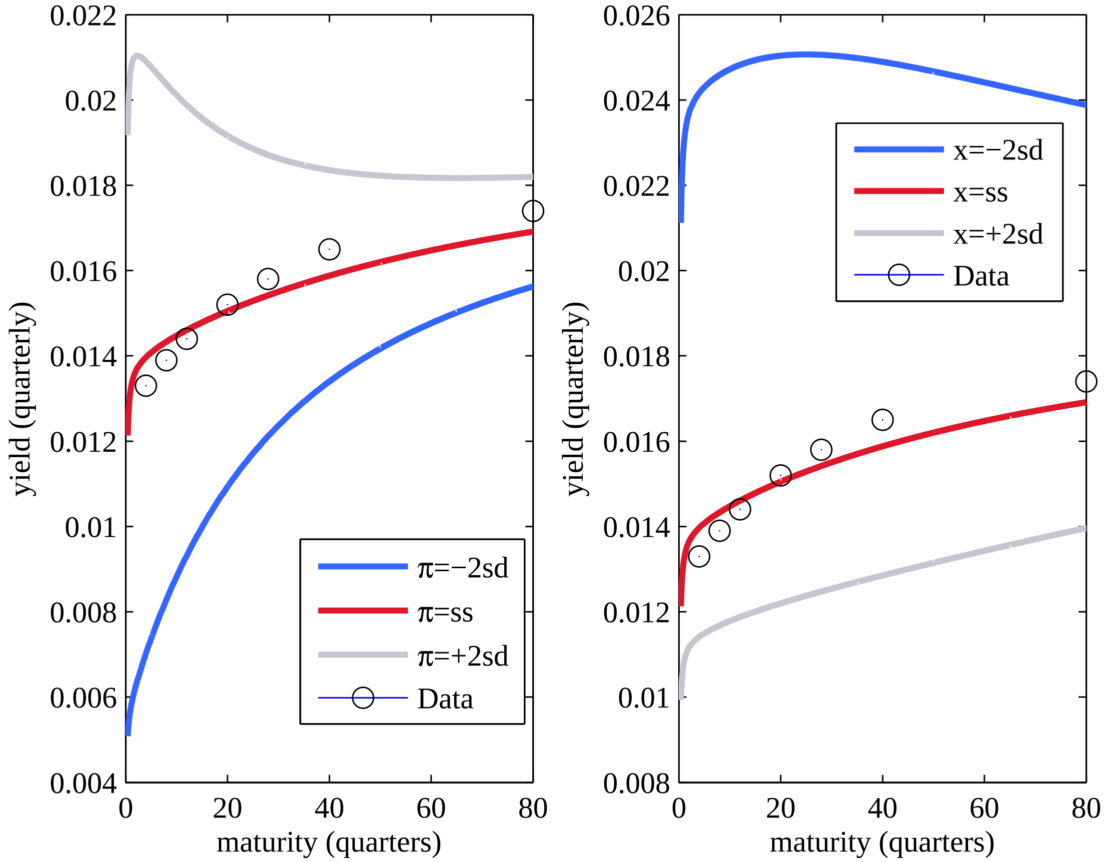
NOTES: The left-hand panel shows the conditional covariance between the stochastic discount factor and real bond returns, across the state space (red, blue, and gray lines correspond to 80, 20, and 4 quarters, respectively). The right-hand panel shows the unconditional covariance across maturities.

FIGURE 4. The Long-Term Real Yield Curve and Volatility of Forward Contracts



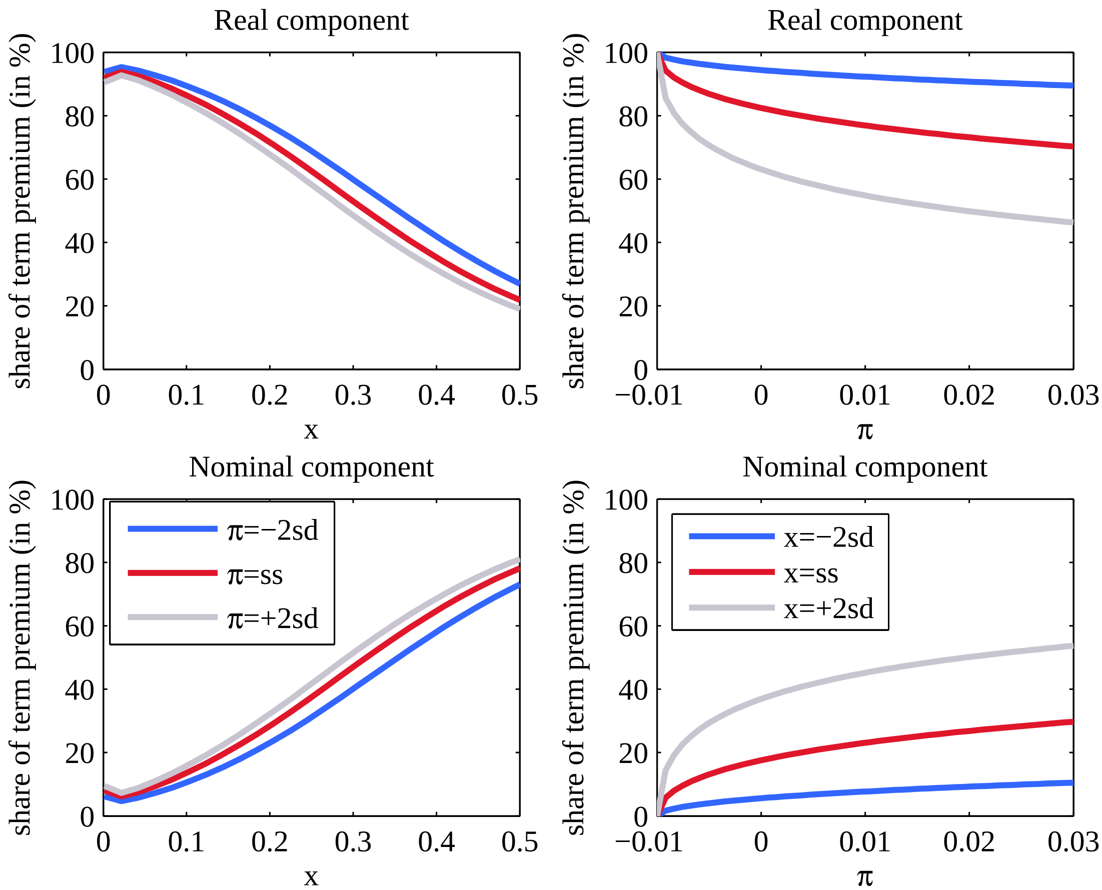
NOTES: The left panel shows the real yield curve conditional on different values of the endogenous state variable x. The blue (gray) line represents the real yield curve when x is 2 standard deviations below (above) its mean. The red line represents the real yield curve when x is at its stochastic steady state. The right panel is the standard deviation of 10 forward contracts, starting with the contract for $1q \to 40q$, continuing with $40q \to 80q$, $80 \to 120$, $120 \to 160$, and so on, until $280 \to 320$ in the last bar.

FIGURE 5. The Nominal Term Structure: Exogenous Inflation Case



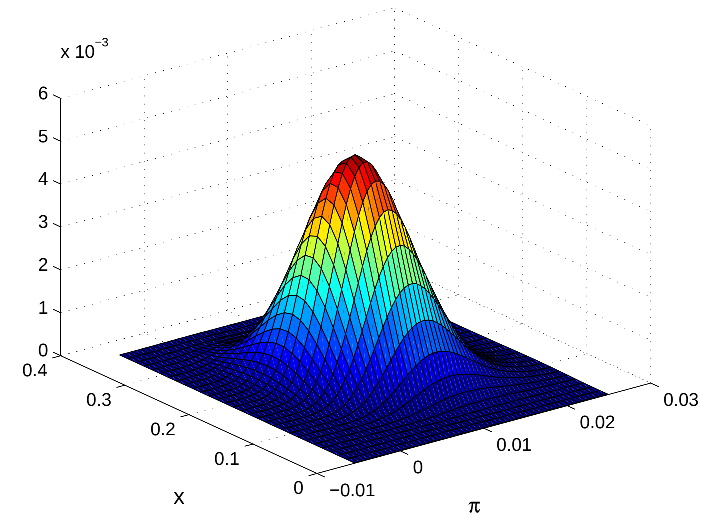
NOTES: This figure shows the nominal term structure of interest rates in the exogenous inflation case. In the left-hand panel, I set x to its unconditional mean, and I show the yield curve for three different values of π . The red line is the nominal term structure when π is at its unconditional mean level; the gray (blue) line is when π is two standard deviations above (below) its unconditional mean level. In the right-hand panel, I set π to its unconditional mean, and I show the yield curve for three different values of x . The red line is the mean x; the gray (blue) line is for x two standard deviations above (below) $x's$ mean. By definition, the red line is the same in both panels. Data from nominal yields are from Table 1, full sample.

FIGURE 6. Decomposition of 80-Quarters' Nominal Yield

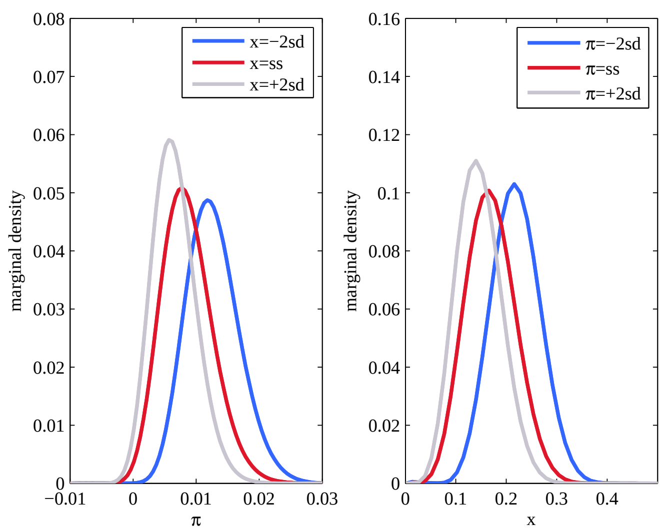


NOTES: This figure shows a decomposition of a 20-year nominal bond term premia. Details of the term premia are in equation (27). The left-hand panel (both upper and lower) display the real and nominal components over the x state space. The three lines represent difference levels for the other state variable, π. The red line is when π is at its unconditional mean; the blue (gray) line is when π is two standard deviations below (above) the steady state. The right-hand panel (both upper and lower) display the real and nominal components over the π state pace. The three lines represent difference levels for the other state variable, x. The red line is when x is at its unconditional mean; the blue (gray) line is when x is two standard deviations below (above) the steady-state level.

FIGURE 7. Invariant Distribution (x, π): Exogenous Inflation



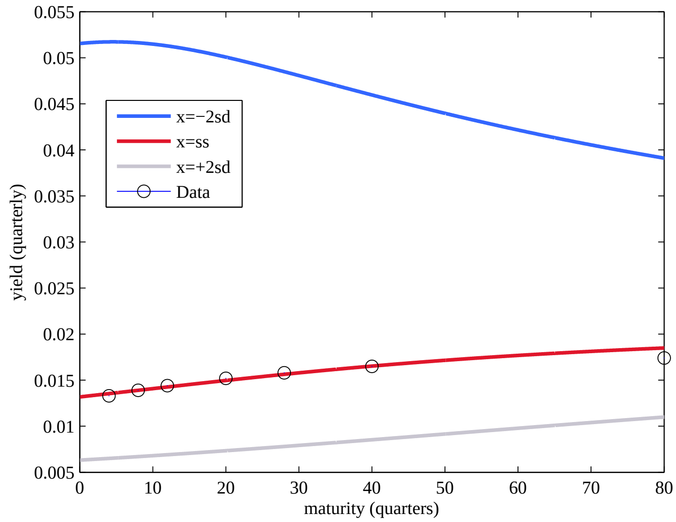
(a) Invariant distribution of (x, π)



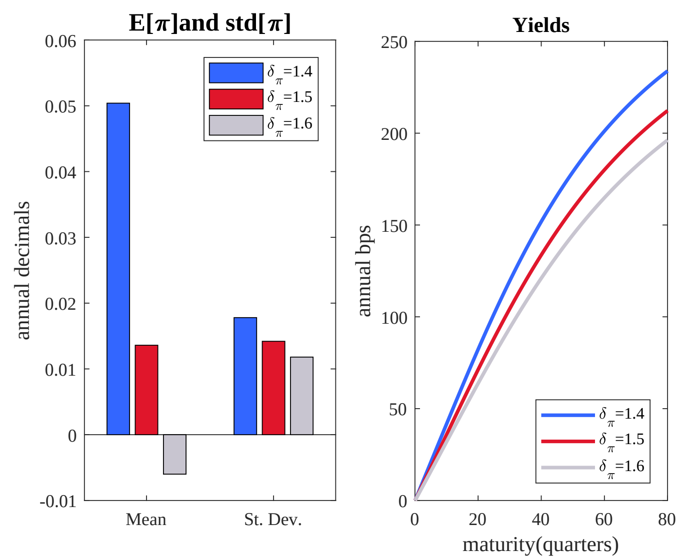
(b) Marginal distributions

NOTES: (a) shows the invariant bi-distribution in (x, π). (b) depicts the marginal distributions. In (b), the left-hand panel shows the marginal invariant distribution for π, for different levels of x: when x is 2 standard deviations below the mean (blue), when it is at its mean (red), and when it is 2 standard deviations above the mean (gray). Similarly, the right-hand panel in (b) illustrates the marginal invariant distributions for x, for different values of π. Marginal distributions are computed by integrating the bivariate mass accordingly.

FIGURE 8. The Nominal Term Structure: Endogenous Inflation Case



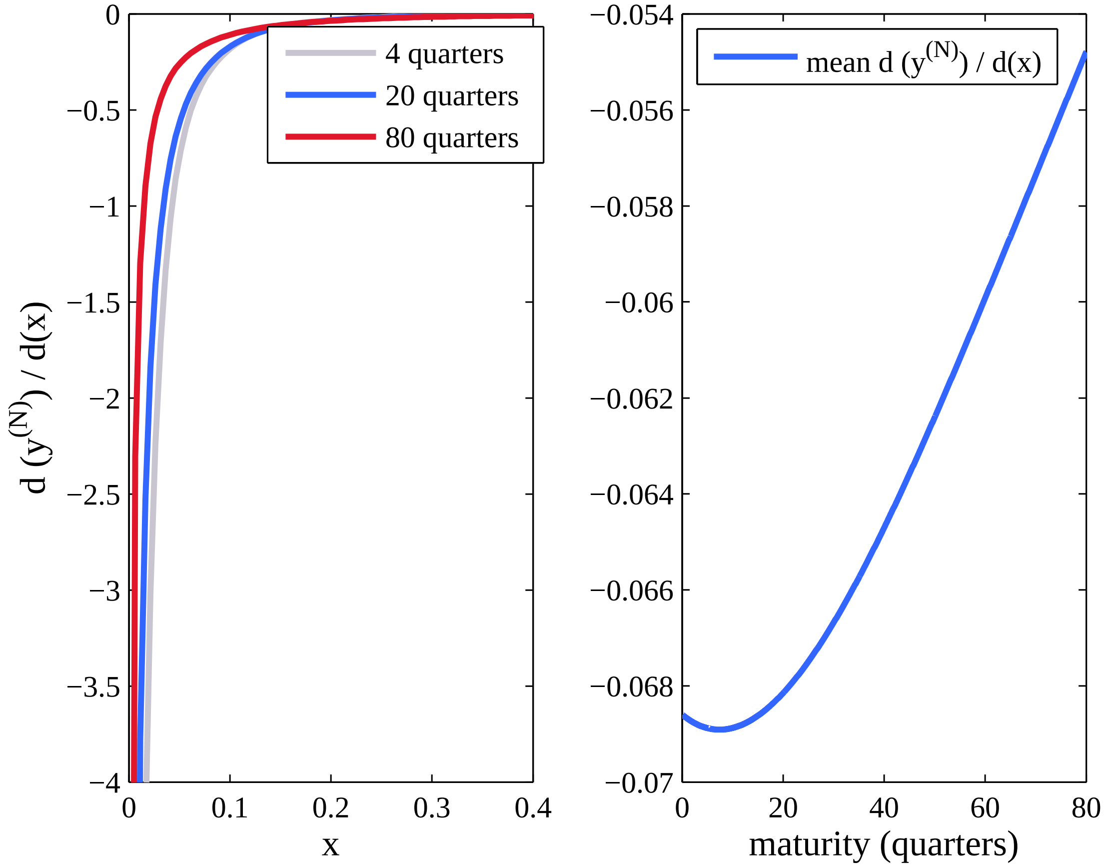
(a) Nominal yield curve: endogenous π with $\delta_\pi = 1.5$



(b) Mean π, std π, and normalized yields for different δ_π

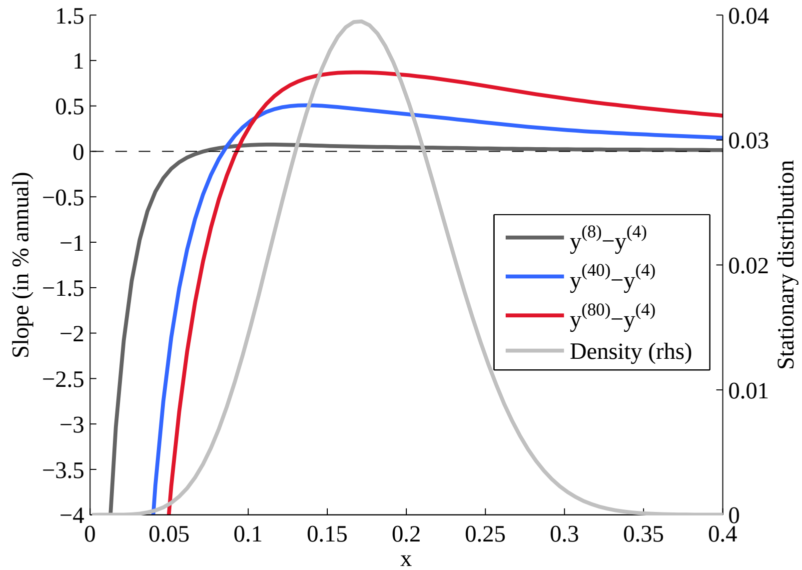
NOTES: (a) shows the nominal term structure in the endogenous inflation case. The red line is when x is at the steady state; the blue (gray) line is when x is two standard deviations below (above) the mean. In (b), the left panel shows how the mean and volatility of inflation for different Taylor coefficients δ_π, and right panel shows the slope of the nominal term structure for different Taylor coefficients δ_π.

FIGURE 9. Average Sensitivity Real Bond Yields in the Model: $E\left[\frac{dy_t^{(T)}}{dx}\right]$

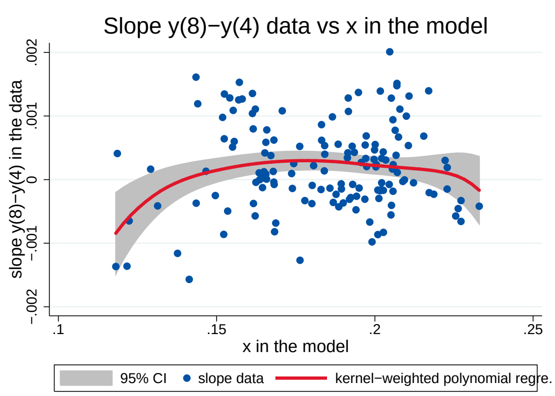


NOTES: The left panel shows the derivative $y_x'^{(T)}$ across the state space (red, blue, and gray lines correspond to 80, 20, and 4 quarters). The right panel shows the unconditional mean across maturities.

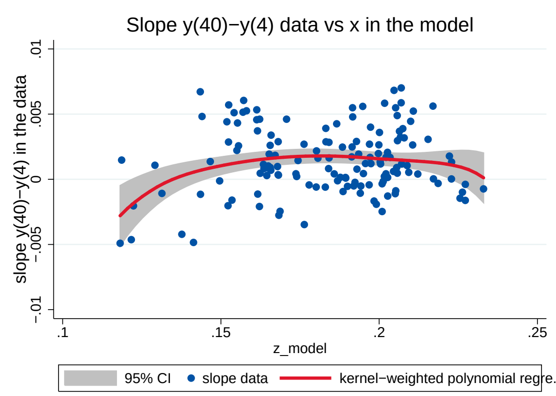
FIGURE 10. Term Structure Slope



(a) Slope in the model



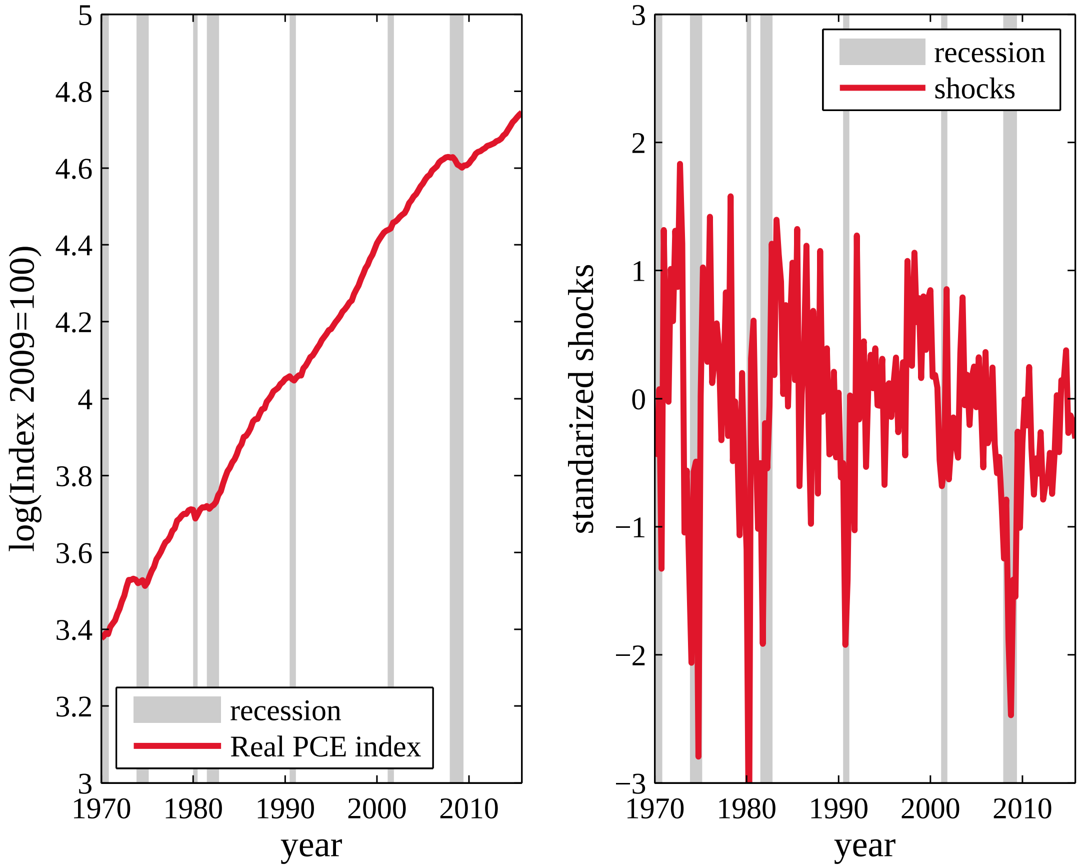
(b) Slope $y^{(8)} - y^{(4)}$ in the data and x in the model



(c) Slope $y^{(40)} - y^{(4)}$ in the data and x in the model

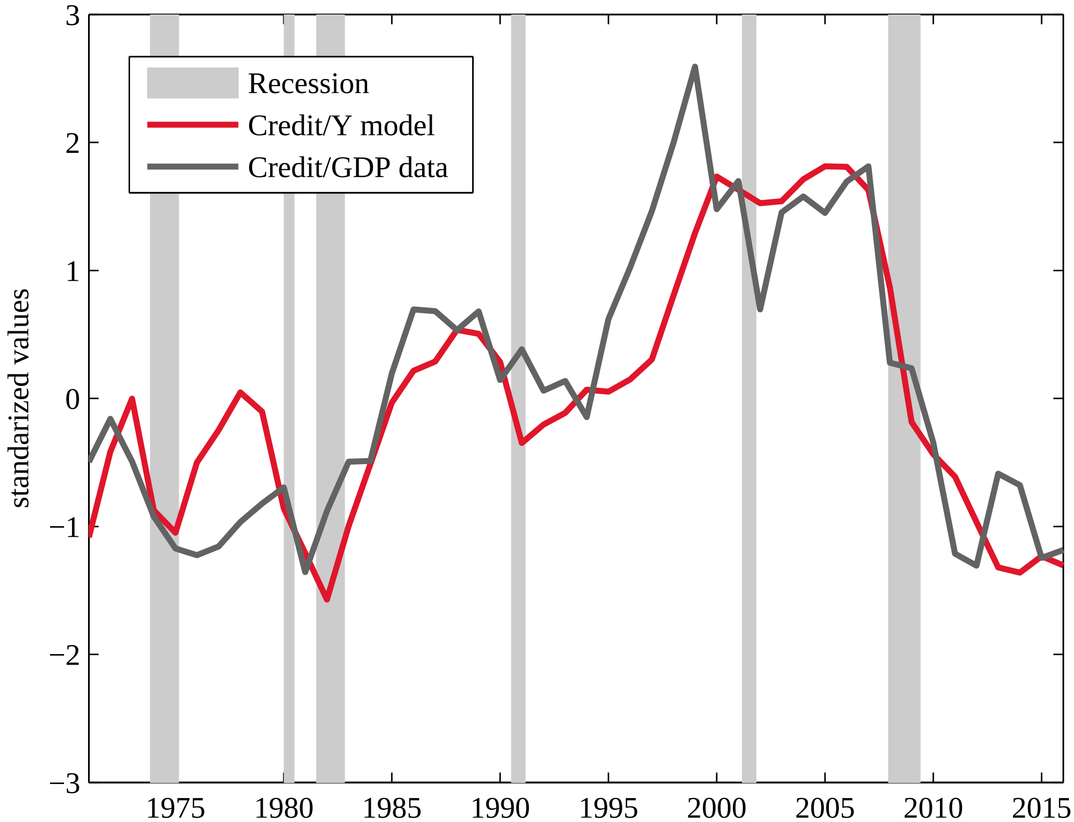
NOTES: (a) shows the slope of the real term structure in the model. The red line is the spread of an 80-quarter yield minus a 4-quarter yield. The blue line is the spread of a 40-quarter yield minus a 4-quarter yield. The black line is the spread of an 8-quarter yield minus a 4-quarter yield. The gray line is the invariant distribution. (b) and (c) show $y^{(8)} - y^{(4)}$ and $y^{(40)} - y^{(4)}$ from the data and x predicted by the model. The red line is the kernel-weighted local polynomial regression. I use an Epanechnikov kernel function.

FIGURE 11. Personal Consumption Expenditure Shocks and Series



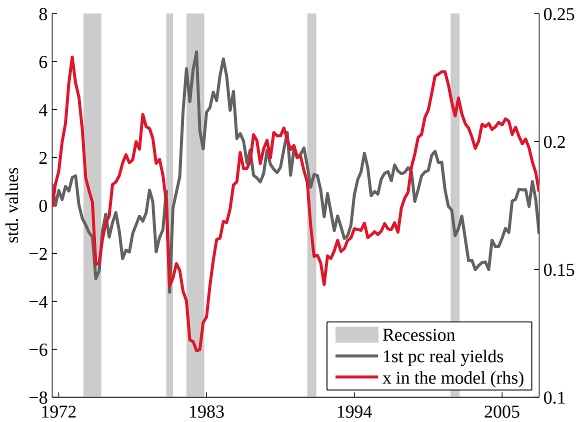
NOTES: The left panel shows the log index of personal consumption expenditures from NIPA Table 2.3.3. The right panel shows the shocks; following equation (34), $\Delta \log y_t - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta = \sigma\left[W_{1,t+\Delta} - W_{1,t}\right]$.

FIGURE 12. Business Cycle Analysis: Credit/GDP Data vs Credit/Y model

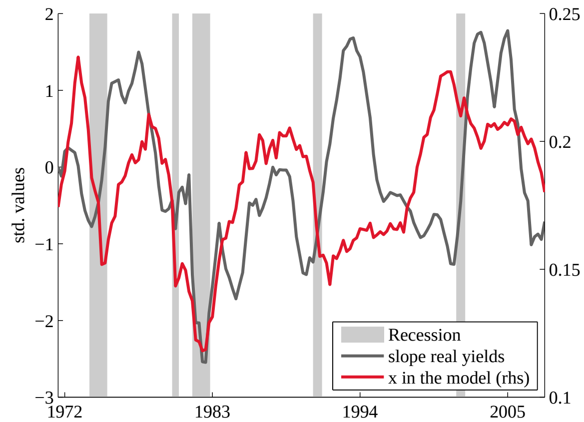


NOTES: The red line shows the implications for credit over total equity in the model after introducing the macro shocks in Figure 11. I start the economy from the stochastic steady state in 1971:Q3 . The black line is the fluctuations in total credit to the private sector over GDP in the U.S. (source: The World Bank).

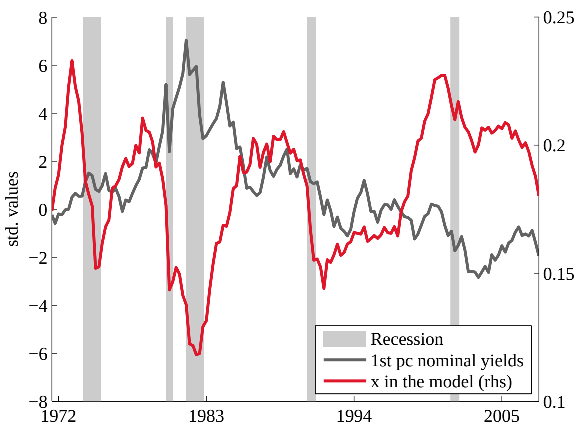
FIGURE 13. Model Implied x and Yield Curve Data



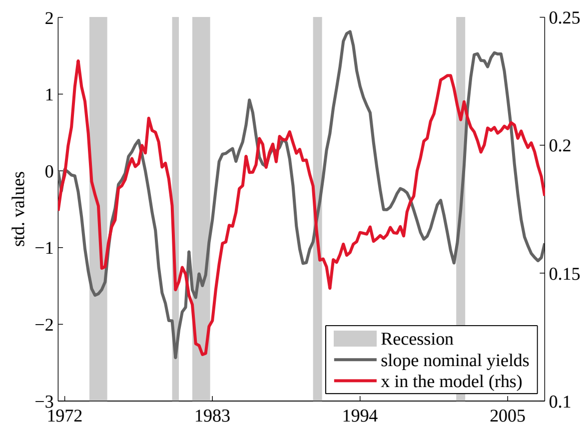
(a) x and the 1pc of real yields



(b) x and the slope of real yields



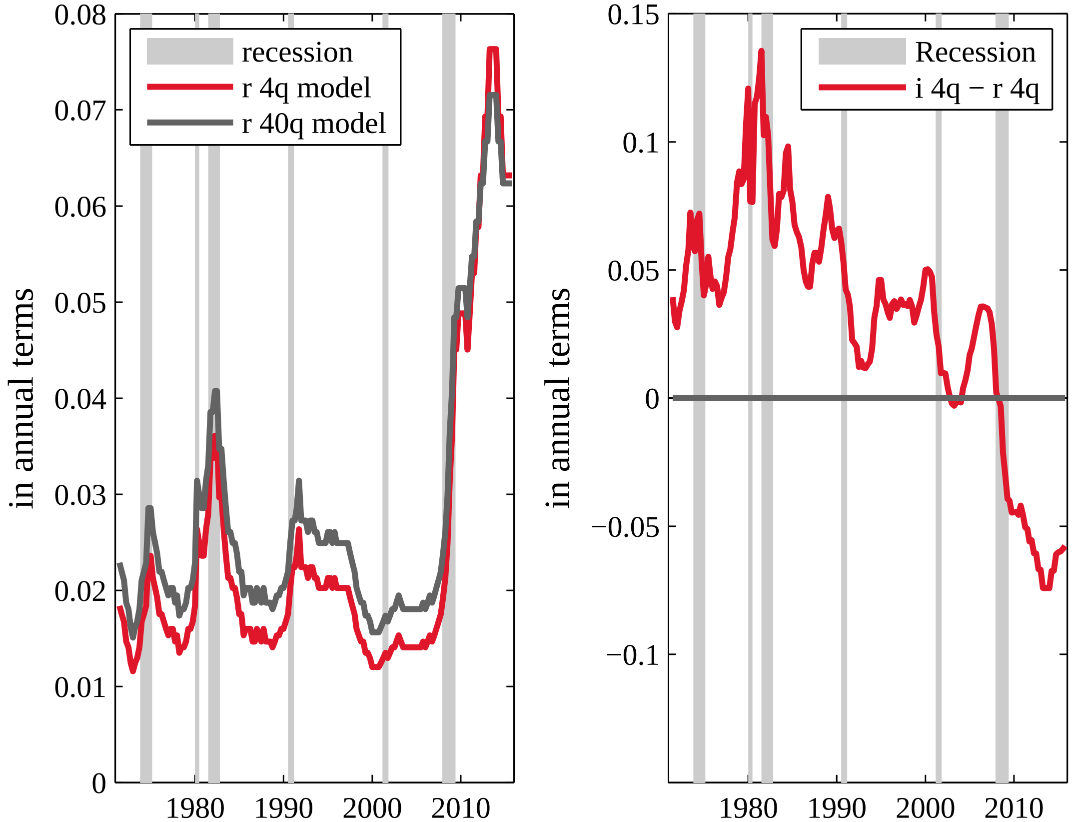
(c) x and the 1pc of nominal yields



(d) x and the slope of nominal yields

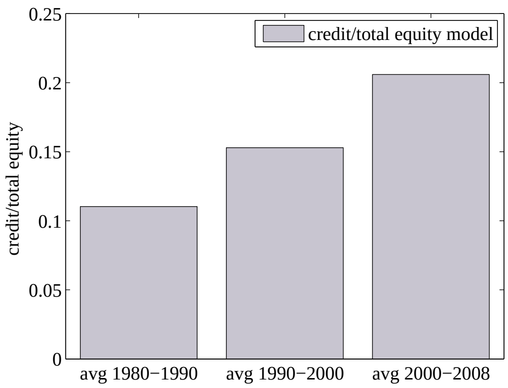
NOTES: Variable x is the endogenous state variable in the model, after feeding the sequence of macro shocks reported in Figure 11, and explained in Section 7. The first principal component of nominal and real yields is computed over the yields considered in Table 1. The slope is the 10-year minus 1-year yield (i.e. ,40 quarters minus 4 quarters), and I compute the annual average of this spread each quarter.

FIGURE 14. Puzzle I: Spike in Real Rates in the Great Recession

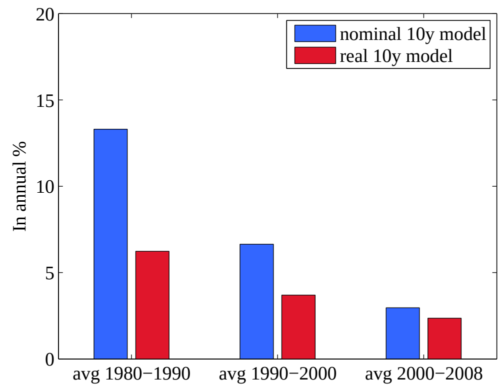


NOTES: The left panel shows the 40-quarter (10-year) yield in black, and the 4-quarter (1-year) in red, predicted by the model when I feed the series of macroeconomic shocks reported in Figure 11. The right panel shows the result of subtracting the model-implied 4-quarter real rate from the - quarter nominal rate in the data. That is, this implies a proxy for implied inflation expectations.

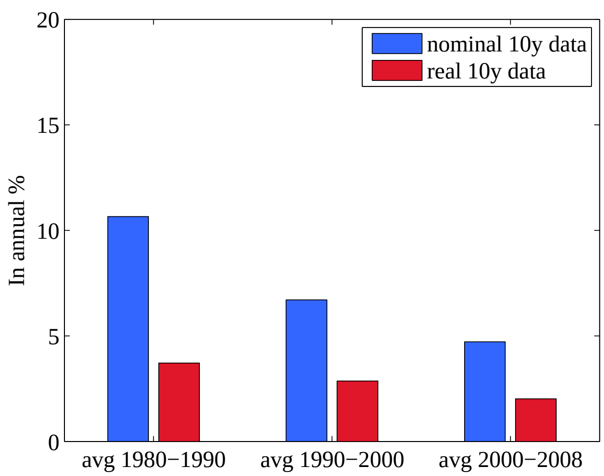
FIGURE 15. Puzzle II: Secular Decline in Long-Term Rates



(a) Credit/Total Equity: Model



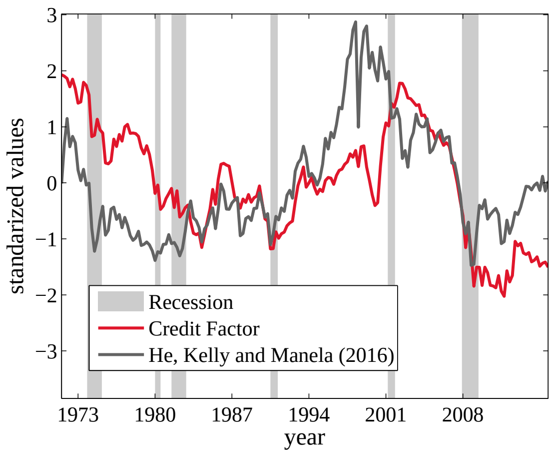
(b) Real and Nominal Long-Term Rates: Model



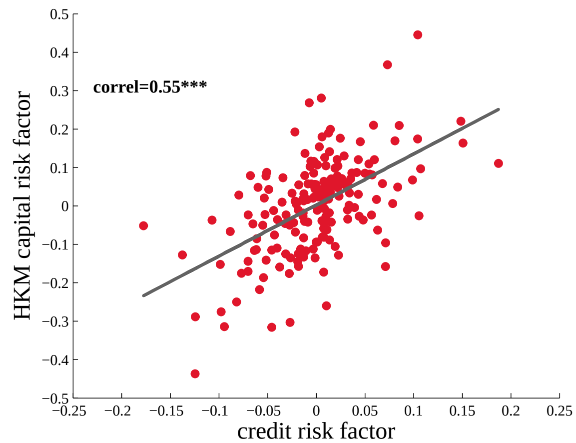
(c) Real and Nominal Long-Term Rates: Data

NOTES: Panels (a) and (b) show the model's predictions after feeding the macro shocks reported in Figure 11, when analyzing the transitional dynamics from 2 standard deviations below the mean of x. Panel (a) shows total credit/total equity in the model. Panel (b) shows the implications for 10-year nominal and real rate in the model. Panel (c) shows 10-year nominal and real rates in the data.

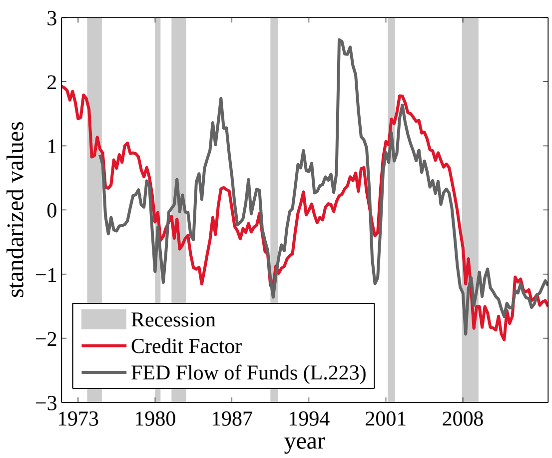
FIGURE 16. Comparing the Credit Factor



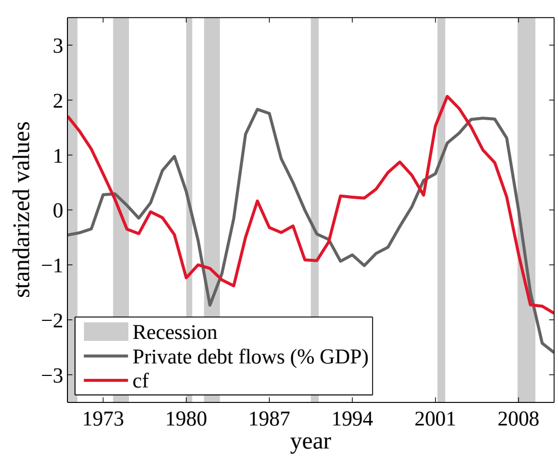
(a) cf and He, Kelly and Manela (2016) factor: Levels



(b) cf and He, Kelly and Manela (2016) factor: Shocks



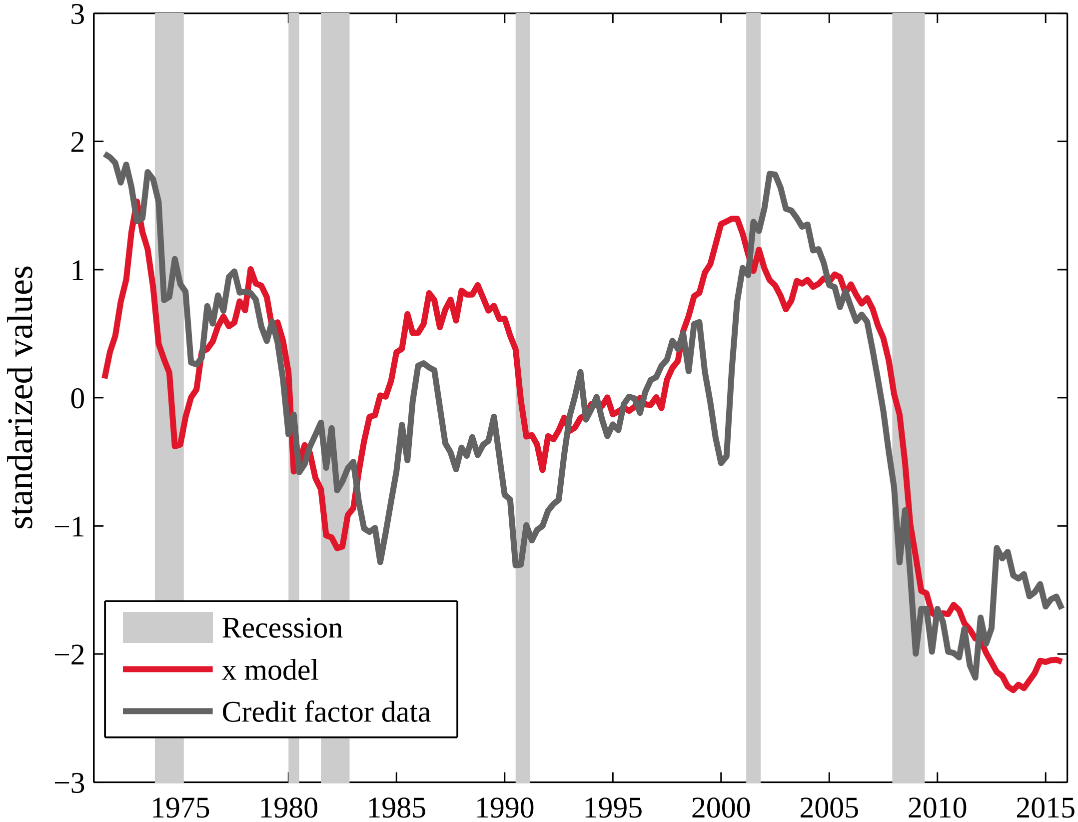
(c) cf and cf from Flow of Funds



(d) cf and debt flows/GDP from Philippon (2015) (annual)

NOTES: Variable Credit Factor (cf), displayed in all panels, is the market value of net worth of SIC codes 60-64 over total net worth (source: CRSP). I do a rolling linear detrending each quarter to remove the persistent component. The availability of cf in CRSP is from 1926, although the figure shows since 1971:Q3 to be consistent with the yields. Panels (a) and (b) compare with the factor used in He et al. (2016). Panel (c) compares with the data in the Flow of Funds, Table L.223. Panel (d) compares, at annual frequency, with the flow of intermediated assets in the financial sector (source: Philippon (2015)).

FIGURE 17. Credit Factor vs x in Model



NOTES: This figure shows the implications for x in the model, after introducing the macro shocks in Figure 11 . The black line is cf defined in Section 7, and displayed in Figure 16.

9 Bibliography

Abrahams, Michael, Tobias Adrian, Richard K. Crump, Emanuel Moench, and Rui Yu, "Decomposing real and nominal yield curves," *Journal of Monetary Economics*, 2016, *84*, 182 – 200.

Ang, Andrew and Monika Piazzesi, "A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables," *Journal of Monetary Economics*, 2003, *50* (4), 745–787.

Atkeson, Andrew and Masao Ogaki, "Wealth-varying intertemporal elasticities of substitution: Evidence from panel and aggregate data," *Journal of Monetary Economics*, 1996, *38* (3), 507 – 534.

Backus, David, Mikhail Chernov, and Stanley E. Zin, "Identifying Taylor Rules in Macro-Finance Models," Working Paper 19360, National Bureau of Economic Research August 2016.

_ , Nina Boyarchenko, and Mikhail Chernov, "Term Structures of Asset Prices and Returns," Forthcoming Journal of Financial Economics, UCLA 2017.

Bansal, Ravi and Amir Yaron, "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *The Journal of Finance*, 2004, *59* (4), 1481–1509.

_ and Ivan Shaliastovich, "A Long-Run Risks Explanation of Predictability Puzzles in Bond and Currency Markets," *The Review of Financial Studies*, 2013, *26* (1), 1.

Barro, Robert J., Jesús Fernández-Villaverde, Oren Levintal, and Andrew Mollerus, "Safe Assets," Working Paper 20652, National Bureau of Economic Research October 2017.

Bauer, Michael D. and Glenn D. Rudebusch, "Resolving the Spanning Puzzle in Macro-Finance Term Structure Models*," *Review of Finance*, 2017, *21* (2), 511–553.

Bernanke, Ben, Carol C. Bertaut, Laurie Pounder Demarco, and Steven B. Kamin, "International capital flows and the returns to safe assets in the United States, 2003-2007," International Finance Discussion Papers 1014, Board of Governors of the Federal Reserve System (U.S.) 2011.

Bhamra, Harjoat S. and Raman Uppal, "The Effect of Introducing a Non-Redundant Derivative on the Volatility of Stock-Market Returns When Agents Differ in Risk Aversion," *The Review of Financial Studies*, 2009, *22* (6), 2303–2330.

Blanchard, Olivier, "Debt, Deficits, and Finite Horizons," *Journal of Political Economy*, 1985, *93* (2), 223–47.

Boyd, John P., *Chebyshev and Fourier Spectral Methods* Dover Books on Mathematics, second ed., Mineola, NY: Dover Publications, 2001.

Caballero, Ricardo J., Emmanuel Farhi, and Pierre-Olivier Gourinchas, "An Equilibrium Model of "Global Imbalances" and Low Interest Rates," *American Economic Review*, March 2008, *98* (1), 358–93.

Campbell, John Y. and John Cochrane, "Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, April 1999, *107* (2), 205–251.

_ , Robert J. Shiller, and Luis M. Viceira, "Understanding Inflation-Indexed Bond Markets," *Brookings Papers on Economic Activity*, 2009, *2009*, 79–120.

Chan, Yeung Lewis and Leonid Kogan, "Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices," *Journal of Political Economy*, 2002, *110* (6), 1255–1285.

Chernov, Mikhail and Philippe Mueller, "The term structure of inflation expectations," *Journal of Financial Economics*, 2012, *106* (2), 367 – 394.

Constantinides, George and Darrell Duffie, "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 1996, *104* (2), 219–40.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 1985, *53* (2), 385–407.

Di Tella, Sebastian, "Uncertainty Shocks and Balance Sheet Recessions," *Forthcoming at the Journal of Political Economy*, 2017.

__ **and Pablo Kurlat**, "Why are Banks Exposed to Monetary Policy?," *Working paper, Stanford University*, 2017.

Dou, Wiston W., "Embrace or Fear Uncertainty: Growth Options, Limited Risk Sharing, and Asset Prices," Working Paper, The Wharton School 2017.

Drechsler, Itamar, Alexi Savov, and Philipp Schnabl, "A Model of Monetary Policy and Risk Premia," *Forthcoming The Journal of Finance*, 2017.

Duffie, Darrel and Pierre-Louis Lions, "PDE solutions of stochastic differential utility," *Journal of Mathematical Economics*, 1992, *21* (6), 577 – 606.

Duffie, Darrell and Larry G Epstein, "Stochastic Differential Utility," *Econometrica*, March 1992, *60* (2), 353–94.

Dumas, Bernard, "Two-Person Dynamic Equilibrium in the Capital Market," *Review of Financial Studies*, 1989, *2* (2), 157–188.

Gallmeyer, Michael F., Burton Hollifield, Francisco J. Palomino, and Stanley E. Zin, "Arbitrage-free bond pricing with dynamic macroeconomic models," *Review*, 2007, (Jul), 305–326.

Gârleanu, Nicolae and Stravos Panageas, "Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing," *Journal of Political Economy*, 2015, *123* (3), 670–685.

Greenwood, Robin and Dimitri Vayanos, "Bond Supply and Excess Bond Returns," *Review of Financial Studies*, 2014, *27* (3), 663–713.

Gürkaynak, Refet S. and Jonathan H. Wright, "Macroeconomics and the Term Structure," *Journal of Economic Literature*, June 2012, *50* (2), 331–67.

__ **, Brian Sack, and Jonathan H. Wright**, "The U.S. Treasury yield curve: 1961 to the present," *Journal of Monetary Economics*, 2007, *54* (8), 2291 – 2304.

__ **,** __ **, and** __ , "The TIPS Yield Curve and Inflation Compensation," *American Economic Journal: Macroeconomics*, January 2010, *2* (1), 70–92.

Guvenen, Fatih, "Reconciling conflicting evidence on the elasticity of intertemporal substitution: A macroeconomic perspective," *Journal of Monetary Economics*, 2006, *53* (7), 1451 – 1472.

— , "A Parsimonious Macroeconomic Model for Asset Pricing," *Econometrica*, 2009, *77* (6), 1711–1750.

Haddad, Valentin and David Sraer, "The Banking View of Bond Risk Premia," *Working paper, UCLA Anderson*, 2015.

Hall, Robert, "The Role of the Growth of Risk-Averse Wealth in the Decline of the Safe Real Interest Rate," *Working paper, Stanford University*, 2017.

He, Zhiguo, Bryan Kelly, and Asaf Manela, "Intermediary Asset Pricing: New Evidence from Many Asset Classes," Working Paper 21920, National Bureau of Economic Research January 2016.

Karlin, S. and H. M. Taylor, *A Second Course in Stochastic Processes*, Academic Press, 1981.

King, Robert G. and Sergio T. Rebelo, "Transitional Dynamics and Economic Growth in the Neoclassical Model," *The American Economic Review*, 1993, *83* (4), 908–931.

Krishnamurthy, Arvind and Annette Vissing-Jorgensen, "The Effects of Quantitative Easing on Interest Rates: Channels and Implications for Policy," *Brookings Papers on Economic Activity*, 2011, *42* (2 (Fall)), 215–287.

Litterman, Robert and José Sheinkman, "Common factors affecting bond returns," *The Journal of Fixed Income*, 1991, *1* (1).

Longstaff, Francis A., "A nonlinear general equilibrium model of the term structure of interest rates," *Journal of Financial Economics*, 1989, *23* (2), 195 – 224.

— **and Jiang Wang**, "Asset Pricing and the Credit Market," *Review of Financial Studies*, 2012, *25* (11), 3169–3215.

Philippon, Thomas, "Has the US Finance Industry Become Less Efficient? On the Theory and Measurement of Financial Intermediation," *American Economic Review*, April 2015, *105* (4), 1408–38.

Piazzesi, Monika and Martin Schneider, "Equilibrium Yield Curves," Working Paper 12609, National Bureau of Economic Research October 2006.

Rudebusch, Glenn D. and Eric T. Swanson, "Examining the bond premium puzzle with a DSGE model," *Journal of Monetary Economics*, 2008, *55* (Supplement), S111 – S126. Contributions to Macroeconomics in Honor of John Taylor.

— **and** — , "The Bond Premium in a DSGE Model with Long-Run Real and Nominal Risks," *American Economic Journal: Macroeconomics*, 2012, *4* (1), 105–143.

Santos, Tano and Pietro Veronesi, "Habits and Leverage," Working Paper 22905, National Bureau of Economic Research December 2016.

Schneider, Andres, "Endogenous and Exogenous Risk Premia," *Working Paper, UCLA*, 2017.

Schroder, Mark and Costis Skiadas, "Optimal Consumption and Portfolio Selection with Stochastic Differential Utility," *Journal of Economic Theory*, 1999, *89* (1), 68 – 126.

Silva, Dejanir, "The Risk Channel of Unconventional Monetary Policy," *Working Paper, MIT*, 2016.

Taylor, John B., "Discretion versus policy rules in practice," *Carnegie-Rochester Conference Series on Public Policy*, 1993, *39*, 195 – 214.

Trefethen, Lloyd N., *Spectral Methods in MatLab*, Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2000.

Wachter, Jessica A., "A consumption-based model of the term structure of interest rates," *Journal of Financial Economics*, 2006, *79* (2), 365 – 399.

Wang, Jiang, "The term structure of interest rates in a pure exchange economy with heterogeneous investors," *Journal of Financial Economics*, 1996, *41* (1), 75 – 110.

10 Appendix

Proof proposition (law of motion for x). The law of motion follows by applying Ito's lemma in (9).

$$\frac{dx_t}{x_t} = \frac{dn_{A,t}}{n_{A,t}} - \frac{dq_t}{q_t} + \left(\frac{dq_t}{q_t}\right)^2 - \left(\frac{dq_t}{q_t}\right)\left(\frac{dn_{A,t}}{n_{A,t}}\right), \tag{39}$$

where the aggregate wealth for A-type investors can be computed as $n_{A,t} = \overline{x}\int_{-\infty}^{t} \varphi e^{-\varphi(t-\widetilde{u})} n_{A,\widetilde{u}} d\widetilde{u}$, and $q_t = n_{A,t} + n_{B,t}$. Then

$$\frac{dq_t}{q_t} = x_t \frac{dn_{A,t}}{n_{A,t}} + (1-x_t)\frac{dn_{B,t}}{n_{B,t}},$$

so, the terms in (39) are

$$\begin{aligned}
\frac{dn_{A,t}}{n_{A,t}} - \frac{dq_t}{q_t} &= (1-x_t)\left(\frac{dn_{A,t}}{n_{A,t}} - \frac{dn_{B,t}}{n_{B,t}}\right), \\
\left(\frac{dq_t}{q_t}\right)\left(\frac{dq_t}{q_t} - \left(\frac{dn_{A,t}}{n_{A,t}}\right)\right) &= \left(\frac{dq_t}{q_t}\right)\left((1-x)\left(\frac{dn_{B,t}}{n_{B,t}} - \frac{dn_{A,t}}{n_{A,t}}\right)\right), \\
&= \sigma_{q1}^2\left(x_t\alpha_{A,t} + (1-x_t)\alpha_{B,t}\right)(1-x)\left(\alpha_{B,t} - \alpha_{A,t}\right) \\
&= \sigma_{q1}^2 (1-x)\left(\alpha_{B,t} - \alpha_{A,t}\right)
\end{aligned}$$

where the last step follows for market clearing for shares. Using Itô's lemma in $n_{A,t}$ and $n_{B,t}$

$$\begin{aligned}
\frac{dn_{A,t}}{n_{A,t}} &= \left[r_t + \varphi - \frac{c_{A,t}}{n_{A,t}} + \alpha_{A,t}\left(\mu_{q,t} - r_t\right)\right]dt + \alpha_{A,t}\sigma_{q,t}dW_{1,t} + \varphi\left(\frac{\overline{x}}{x_t}\frac{\widehat{e}_t}{pd_t} - 1\right)dt, \\
\frac{dn_{B,t}}{n_{B,t}} &= \left[r_t + \varphi - \frac{c_{B,t}}{n_{B,t}} + \alpha_{B,t}\left(\mu_{q,t} - r_t\right)\right]dt + \alpha_{B,t}\sigma_{q,t}dW_{1,t} + \varphi\left(\left(\frac{1-\overline{x}}{1-x_t}\right)\frac{\widehat{e}_t}{pd_t} - 1\right)dt,
\end{aligned}$$

Then

$$\begin{aligned}
\sigma_x &= x(1-x_t)\left(\frac{dn_{A,t}}{n_{A,t}} - \frac{dn_{B,t}}{n_{B,t}}\right) = x(1-x_t)\left(\alpha_{A,t} - \alpha_{B,t}\right), \\
\mu_{x,t} &= x_t(1-x_t)\left(\frac{c_{B,t}}{n_{B,t}} - \frac{c_{A,t}}{n_{A,t}} + \left(\alpha_{A,t} - \alpha_{B,t}\right)\left(\mu_{q,t} - r_t - \sigma_{q1,t}^2\right)\right) + x_t(1-x_t)\frac{\varphi\widehat{e}_t}{pd_t}\left(\frac{\overline{x}}{x_t} - \frac{1-\overline{x}}{1-x_t}\right),
\end{aligned}$$

and then $x_t(1-x_t)\left(\frac{\overline{x}}{x_t} - \frac{1-\overline{x}}{1-x_t}\right) = \left(\overline{x}(1-x_t) - (1-\overline{x})x_t\right) = \left(\overline{x} - \overline{x}x_t - x_t + \overline{x}x_t\right) = \left(\overline{x} - x_t\right)$.
■

Proof proposition (leverage and risk sharing). As stated in FOC, the portfolio share of A-type agents is

$$\alpha_A = \frac{\mu_q - r}{\gamma_A\sigma_q^2} + \left(\frac{1-\gamma_A}{1-\psi_i}\right)\frac{\sigma_{\xi A}}{\gamma_A\sigma_q},$$

where, notice,

$$\sigma_{\xi A} = \frac{\xi_{x,A}}{\xi_A}x(1-x)\left(\alpha_B - \alpha_A\right)\sigma_q$$

which means

$$\begin{aligned}\frac{\mu_{q,t}-r_t}{\sigma_q^2} &= \gamma_A\alpha_A-\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\sigma_{\xi A}}{\sigma_q}\\ &= \gamma_A\alpha_A-\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A}x\left(1-x\right)\left(\alpha_A-\alpha_B\right).\end{aligned}$$

Use market clearing for shares

$$\begin{aligned}x\alpha_A+\left(1-x\right)\alpha_B &= 1\\ \left(1-x\right)\left(\alpha_A-\alpha_B\right) &= \left(\alpha_A-1\right)\end{aligned}$$

means

$$\begin{aligned}\frac{\mu_{q,t}-r_t}{\sigma_q^2} &= \gamma_A\alpha_A-\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A}x\left(\alpha_A-1\right)\\ &= \alpha_A\left[\gamma_A-\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A}x\right]+\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A}x.\end{aligned}$$

So, we can use this in the conditions for B

$$\begin{aligned}\alpha_A\left[\gamma_A-\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A}x\right]+\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A}x+\left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B}x\left(\alpha_A-1\right) &= \gamma_B\alpha_B\\ \alpha_A\left[\gamma_A-\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A}x+\left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B}x\right]+\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A}x-\left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B}x &= \gamma_B\alpha_B.\end{aligned}$$

Define

$$R_t^x=\left[\left(\frac{1-\gamma_A}{1-\psi_A}\right)\frac{\xi_{x,A}}{\xi_A}-\left(\frac{1-\gamma_B}{1-\psi_B}\right)\frac{\xi_{x,B}}{\xi_B}\right]x$$

then

$$\alpha_A\left[\frac{\gamma_A}{\gamma_B}-\frac{R_t^x}{\gamma_B}\right]+\frac{R_t^x}{\gamma_B}=\alpha_B,$$

so

$$\begin{aligned}x\alpha_A+\left(1-x\right)\alpha_B &= 1\\ x\alpha_A+\left(1-x\right)\left[\alpha_A\left[\frac{\gamma_A}{\gamma_B}-\frac{R_t^x}{\gamma_B}\right]+\frac{R_t^x}{\gamma_B}\right] &= 1\\ x\alpha_A+\left(1-x\right)\alpha_A\left[\frac{\gamma_A}{\gamma_B}-\frac{R_t^x}{\gamma_B}\right]+\left(1-x\right)\frac{R_t^x}{\gamma_B} &= 1\\ \alpha_A &= \frac{1-\left(1-x\right)\frac{R_t^x}{\gamma_B}}{x+\left(1-x\right)\left[\frac{\gamma_A}{\gamma_B}-\frac{R_t^x}{\gamma_B}\right]},\end{aligned}$$

so

$$\begin{aligned}\alpha_A-1 &= \frac{1-\left(1-x\right)\frac{R_t^x}{\gamma_B}}{x+\left(1-x\right)\left[\frac{\gamma_A}{\gamma_B}-\frac{R_t^x}{\gamma_B}\right]}-1\\ &= \frac{\left(1-x\right)\left(\gamma_B-\gamma_A\right)}{\gamma_Bx+\left(1-x\right)\left[\gamma_A-R_t^x\right]}.\end{aligned}$$

Notice $\alpha_A > 1 \Leftrightarrow$

$$\begin{aligned} 1-(1-x)\frac{R_t^x}{\gamma_B} &> x+(1-x)\frac{\gamma_A}{\gamma_B}-(1-x)\frac{R_t^x}{\gamma_B} \\ 1-x &> (1-x)\frac{\gamma_A}{\gamma_B} \\ \gamma_B &> \gamma_A \end{aligned}$$

■

Proof proposition (stochastic discount factor). Suppose there is a unique stochastic discount factor m, with a drift given by a process r and diffusion (price of risk) given by κ. Then, the absence of arbitrage implies that

$$\begin{aligned} \mu_q - r_t &= E_t^{\mathbb{P}}\left[\frac{dm_t}{m_t}\frac{dq_t}{q_t}\right] \\ &= \sigma_{q1,t}\kappa_t. \end{aligned} \tag{40}$$

We can solve for κ_t using the agent's FOCs and market clearing-conditions. In particular, I define $\varepsilon_{i,t} = \alpha_{i,t}\sigma_{q1,t}$ as the exposure chosen by agent i to W_1 shocks. The price of this exposure is κ_t. Then,

$$\alpha_{i,t}\left(\mu_q - r_t\right) = \varepsilon_{i,t}\kappa_t.$$

Then the FOC for $\varepsilon_{i,t}$ are

$$\varepsilon_{i,t} = \frac{\kappa_t}{\gamma_i} + \left(\frac{1-\gamma_i}{(1-\psi_i)\gamma_i}\right)\sigma_{\xi i},$$

and using market clearing for shares, I get $x\varepsilon_A + (1-x)\varepsilon_B = \sigma_{q1}$, so

$$\kappa(x) = \frac{\sigma_{q1} - x\left(\frac{1-\gamma_A}{(1-\psi_A)\gamma_A}\right)\sigma_{\xi A} - (1-x)\left(\frac{1-\gamma_B}{(1-\psi_B)\gamma_B}\right)\sigma_{\xi B}}{\frac{x}{\gamma_A}+\frac{1-x}{\gamma_B}}$$

where

$$\sigma_{\xi i} = \frac{\xi'_{ix}}{\xi_i}\sigma_x.$$

Then, the risk free rate follows by the no-arbitrage condition (40). Following expression (10), and incorporating the laws of motion $\frac{dn_{A,t}}{n_{A,t}}, \frac{dn_{B,t}}{n_{B,t}}$, I obtain

$$\begin{aligned} \frac{dq_t}{q_t} = &\left[r_t + \delta + \left(x_t\alpha_{A,t} + (1-x_t)\alpha_{B,t}\right)\left(\mu_{q,t} - r_t\right) - x\frac{c_{A,t}}{n_{A,t}} - (1-x_t)\frac{c_{B,t}}{n_{B,t}}\right]dt \\ &+ \left(x_t\alpha_{A,t} + (1-x_t)\alpha_{B,t}\right)\sigma_{q1,t}dW_{1,t} \\ &+ \left[\varphi\frac{e_t}{q_t} - \varphi\right]dt. \end{aligned}$$

Thus, using market clearing for goods and shares, and then canceling out

$$\begin{aligned} \frac{dq_t}{q_t} &= \left[\mu_{q,t} + \varphi\frac{e_t}{q_t} - \frac{y_t}{q_t}\right]dt + \sigma_{q1,t}dW_{1,t}, \\ \frac{dq_t}{q_t} + \frac{y_t - \varphi e_t}{q_t}dt &= \mu_{q,t}dt + \sigma_{q1,t}dW_{1,t}. \end{aligned}$$

By no-arbitrage, I obtain the expression in equation (19)

$$E^{\mathbb{P}}\left[\frac{dq_t}{q_t}\right]+\frac{y_t-\varphi e_t}{q_t}dt-r_t-\sigma_{q1,t}\kappa_t=0.$$

Using $pd_t = q_t/y_t$, this can be written as an ordinary differential equation in $pd\left(x_t\right)$. That is, using Itô's lemma in the function $pd_t y_t = q_t$, I have $\mu_q = \mu_{pd}\left(x\right)+\mu+\sigma_{pd}\left(x\right)\sigma$ and $\sigma_{q1}\left(x\right) = \sigma_{pd}\left(x\right)+\sigma$. The functions are simply $\mu_{pd}\left(x\right)=\frac{pd'_x}{pd}E\left[dx\right]+\frac{1}{2}\frac{pd''_{xx}}{pd}E\left[dx^2\right]$ and $\sigma_{pd}=\frac{pd'_x}{pd}\sigma_x$. ■

Proof proposition (infinitely lived investor). I first solve for the value function of the representative investment investor in the economy with aggregate endowment (1). To that end, I use the same power form as in (12), together with the first-order condition for consumption. Then, I can substitute to get

$$\mathcal{U}=\frac{c^{1-\gamma}}{1-\gamma}\xi^{\frac{(1-\gamma)\psi}{1-\psi}}, \tag{41}$$

with ξ being a constant (i.e., there are no endogenous fluctuations in the investment opportunity set). I can then use (41) in $0=f\left(\mathcal{U},c\right)+E^{\mathbb{P}}\left[d\mathcal{U}\right]$ to solve for ξ

$$\begin{aligned}
0 &= \frac{\rho}{1-\frac{1}{\psi}}\left(\xi-1\right)+\mu-\frac{\gamma}{2}\sigma^2 \\
\xi &= \left[\frac{\gamma}{2}\sigma^2-\mu\right]\left(\frac{1-\frac{1}{\psi}}{\rho}\right)+1
\end{aligned}$$

The stochastic discount factor, following the martingale approach developed in Schroder and Skiadas (1999), is given by

$$m_t=\exp\left(\int_0^t f'_{\mathcal{U},u}du\right)f'_{c,t}\,, \tag{42}$$

where the derivatives with respect to the value function $\mathcal{U}$ and c are given by

$$\begin{aligned}
f'_c &= \rho c^{-\frac{1}{\psi}}\left(\left(1-\gamma\right)\mathcal{U}\right)^{\frac{\left(\frac{1}{\psi}-\gamma\right)}{1-\gamma}}\,, \\
f'_{\mathcal{U}} &= \frac{\rho}{1-\frac{1}{\psi}}\left(\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\right)c^{1-\frac{1}{\psi}}\left(1-\gamma\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}\mathcal{U}^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}-\frac{\rho\left(1-\gamma\right)}{1-\frac{1}{\psi}}\,,
\end{aligned}$$

so the risk-free rate is

$$r_t=-E^{\mathbb{P}}\left[\frac{dm_t}{m_t}\right]$$

Using Itô's lemma in (42) and computing the expectation yields

$$r_t=\bar{r}\equiv\frac{\rho\left(1-\gamma\right)\psi}{1-\psi}\left[\xi-1\right]+\frac{1}{\psi}\mu-\frac{1}{\psi}\left(\frac{1}{\psi}+1\right)\sigma^2.$$

To compute real bond prices, I use a guess-and-verify procedure. That is, I guess that real bond prices are exponentially affine in the time dimension

$$\begin{aligned}
P^{(T)}\left(t\right) &= \exp\left(A\left(t\right)\right), \\
& \quad \textit{with} \\
P^{(0)}\left(t\right) &= 1,\ \forall t
\end{aligned}$$

where A_t is an unknown function of time. Real bond prices are characterized by the same Cauchy problem as in the main text,

$$\begin{aligned} P_t'^{(T)} &= \bar{r}P \\ A_t' &= r \\ & \quad with \\ A(0) &= 0 \end{aligned} \tag{43}$$

so the ODE (43) is very simple: $A(t) = \bar{r}t$. Then yields $y_t^{(T)} = -\frac{1}{T}\log P_t^{(T)} = \bar{r}\,\forall\,(t,T)$.

Then the price-dividend ratio is characterized by

$$pd_t = \frac{q_t}{y_t} = E_t^{\mathbb{P}}\left[\int_t^{\infty}\frac{m_{t+u}}{m_t}\frac{y_u}{y_t}du\right].$$

The $pd_t = \overline{pd}$ is constant, and it has the standard Gordon growth expression. The partial differential equation characterizing the nominal bond is

$$\begin{aligned} \frac{P_t'(\pi,t)}{P(\pi,t)} &= -\bar{r} - \pi + \frac{P_\pi'(\pi,t)}{P(\pi,t)}\lambda_\pi(\overline{\pi} - \pi_t) + \frac{1}{2}\frac{P_{\pi\pi}''(\pi,t)}{P(\pi,t)}\sigma_\pi^2(\pi - \pi_L) - \frac{P_\pi'(\pi,t)}{P(\pi,t)}\gamma\varphi_{13}\sigma\sigma_\pi\sqrt{\pi - \pi_L}, \\ P(\pi,0) &= 1\,\forall\pi. \end{aligned} \tag{44}$$

I next change variables and assume that $4(\overline{\pi} + \pi_L)\lambda_\pi = \sigma_\pi^2$ to ease calculations[19]

$$z = \pi - \pi_L\,,$$

and notice

$$\begin{aligned} P(\pi) &= P(z + \pi_L)\,, \\ \frac{P_\pi'}{P} &= \frac{P_z'}{P}. \end{aligned}$$

Use the solution

$$P = \widetilde{A}(t)\exp\left(B(t)z + C(t)\sqrt{z}\right)\,, \tag{45}$$

where $\widetilde{A}(t)$ is adjusted for the change in variables. Substituting (45) in (44), functions $\widetilde{A}$, B, and C solve a system of ordinary differential equations. In particular, the solution for B follows a particular case of the Riccati equation:

$$\begin{aligned} B_t' &= \frac{\sigma_\pi^2}{2}B(t)^2 - \lambda_\pi B(t) - 1\,, \\ B(0) &= 0\,, \end{aligned}$$

the function $C(t)$, associated with the $\sqrt{x}$ term solves

$$\begin{aligned} 0 &= -C_t' - \lambda_\pi C(t) - B(t)\gamma\varphi_{13}\sigma\sigma_\pi + \frac{\sigma_\pi^2}{2}B(t)C(t)\,, \\ C(0) &= 0\,, \end{aligned}$$

[19] Notice that when $\pi_L = 0$, this assumption would led to a violation of the so-called Feller condition. Because $\pi_L < 0$, the process is reflected at a point that is below 0.

and the constant

$$\begin{aligned}
0 &= -\frac{\widetilde{A}_t'}{\widetilde{A}} - \bar{r} + \pi_L + \lambda_\pi \left(\overline{\pi} + \pi_L\right) B\left(t\right) - \frac{1}{2} C\left(t\right) \gamma \varphi_{13} \sigma \sigma_\pi + \frac{\sigma_\pi^2}{2} C\left(t\right)^2 \\
\widetilde{A}\left(0\right) &= 1.
\end{aligned}$$

Although the solution for $A(t), B(t)$, and $C(t)$ can be solved in closed form, I omit it in the interest of space(see Longstaff (1989)). However, this system of ODEs can be solved numerically with any standard routine.

■

Evidence for the U.K. In the table below, I report evidence for real and nominal yields in the U.K.. The source of the data is the Bank of England.[20] I use the same criteria adopted for U.S. data and consider the Full Sample the largest available sample for real yields. This includes 1985:Q1-2016:Q4 for the U.K. Short sample I and Short sample II are as in the U.S. data, 2003:Q1-2016:Q4 and 1985:Q1-2008:Q2. I report real rates starting from 3 years, because the shorter maturity available is 2.5 years.

					Maturity (quarters)				
Panel A. Full Sample		4	8	12	20	28	40	80	diff(40-12)
Mean	Nominal	0.0134	0.0136	0.0138	0.0143	0.0147	0.0150	0.0139	0.0012
	Real			0.0044	0.0048	0.0050	0.0053	0.0046	0.0006
St. dev	Nominal	0.0094	0.0089	0.0085	0.0080	0.0077	0.0072	0.0062	-0.0012
	Real			0.0051	0.0047	0.0044	0.0043	0.0040	-0.0011
Panel B. Short Sample I									
Mean	Nominal	0.0053	0.0057	0.0061	0.0071	0.0078	0.0085	0.0096	0.0028
	Real			0.0000	0.0001	0.0001	0.0013	0.0015	0.0013
St. Dev	Nominal	0.0051	0.0048	0.0045	0.0040	0.0036	0.0031	0.0022	-0.0017
	Real			0.0044	0.0038	0.0034	0.0030	0.0024	-0.0014
Panel C. Short Sample II									
Mean	Nominal	0.0177	0.0177	0.0178	0.0179	0.0180	0.0179	0.0160	0.002
	Real			0.0071	0.0072	0.0073	0.0074	0.0067	0.003
St. Dev	Nominal	0.0070	0.0065	0.0062	0.0061	0.0060	0.0059	0.0061	-0.006
	Real			0.0022	0.0021	0.0022	0.0022	0.0024	0.000

As can be seen in the table, the nominal and real term structures share similar properties: They are upward sloping on average up to 40 quarters (10 years) but then the average yield of an 80-quarter bond is smaller than the 40-quarter. Small sample II is the exception, in which on average both real and nominal yield curve are upward sloping. The volatility of long-term rates is smaller than the volatility of short-term yields. Indeed, the volatility of real and nominal yields is very similar in Short Sample I (as in the U.S.).

[20] http://www.bankofengland.co.uk/statistics/pages/yieldcurve/default.aspx

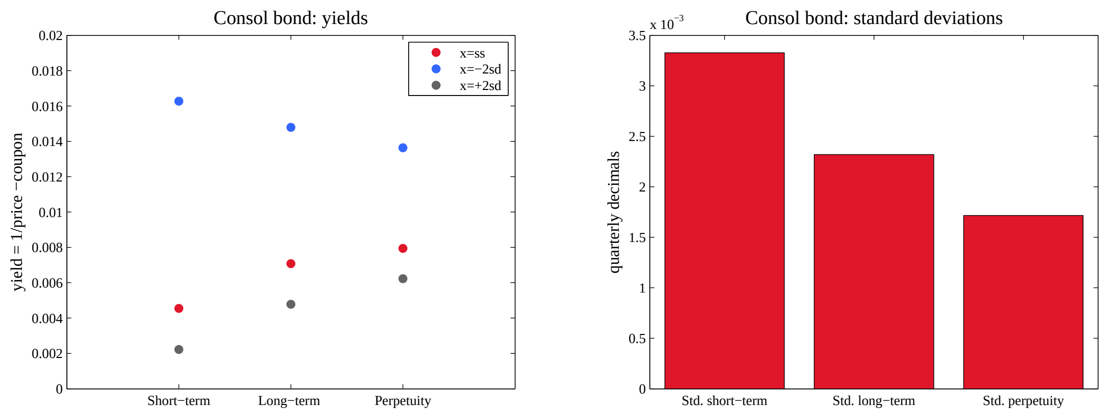
Extension: Consol bond. In this extension, I consider the pricing of a perpetual (real) bond with an exponentially decaying coupon, denoted δ. The purpose is to illustrate the main results in the paper using these alternative financial instruments. To avoid redundancy with the analysis presented above, I include this subsection in the appendix. The price of the consol bond, denoted C_t, is

$$C_t = E_t^{\mathbb{Q}}\left[\int_t^{\infty} e^{-\int_t^s (r_u+\delta)du} ds\right] \equiv C(x_t)$$

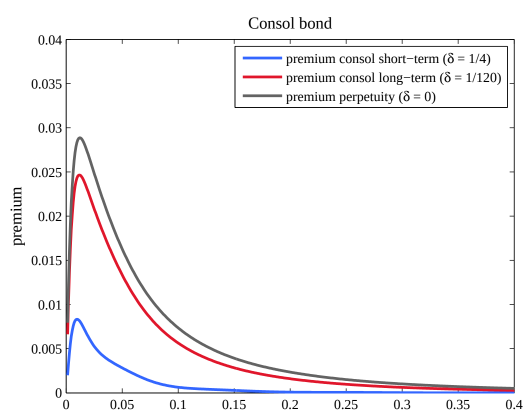
Thus, the yield of the bond is $y_{C,t} = 1/C_t - \delta$. The maturity of the bond is determined by $1/\delta$, and when $\delta = 0$, the bond is a perpetuity. The consol bond is characterized by the following ordinary differential equation:

$$-(r(x)+\delta) + \frac{1}{C} + \frac{C'_x}{C}\mu_x(x) + \frac{1}{2}\frac{C''_{xx}}{C}\sigma_x(x)^2 - \frac{C'_x}{C}\sigma_x(x)\kappa(x) = 0.$$

In the next figure, I show the yield of C_t for $\delta = 1/4 (= \delta_{short-term})$, $\delta = 1/120 (= \delta_{long-term})$, and $\delta = 0 (= \delta_{perpetuity})$. These are proxies for a 4-quarter, 120-quarter, and perpetual zero-coupon bonds. The left panel shows the yield y_C for different levels of x. The right panel shows the standard deviation of the three yields. As in the main text, the term structure is upward sloping and the volatility of yields decreases with maturity.



Lastly, the next figure shows the premium associated with each consol bond (i.e., $cov_t^{\mathbb{P}}\left(\frac{dm}{m}, \frac{dC}{C}\right) = \frac{C'_x}{C}\sigma_x(x)\kappa(x)$). The figure shows that long-term bonds pay an average higher compensation for risk.



Numerical procedure. As mentioned in the text, I use a spectral collocation method based on Chebyshev polynomials of the first kind to solve the problems numerically (model, real yield curve, nominal yield curve, consol bonds, invariant distribution for x, invariant bivariate distribution for x and π). This technique yields a highly accurate global solution.

The solution of the model consists of 4 functions that depend on x_t: two value functions, ξ_A, ξ_B, the valuation of the aggregate earnings and the valuation of the endowment claim. Equilibrium is characterized by a system of nonlinear ordinary differential equations with the 2 HJB equations, the no-arbitrage condition for the endowment claim and the initial earnings, the market clearing conditions for goods ($x\xi_A + (1-x)\xi_B = y/q$), the market clearing condition for shares, and the first order conditions. Real bond prices are characterized by the partial differential equation in (22), where prices depend on two state variables (x and t). Nominal bond prices with exogenous inflation are characterized by the partial differential in (26), where prices depend on three state variables (x, π and t).

The procedure is as follows. Consider a generic function $h(x) : (0,1) \to R$. Then, the function can be written in a polynomial form as

$$h(x) = \sum_{i=0}^{K} a_i \Psi_i(\omega_i(x)) + O(K), \tag{46}$$

where K is the order of the polynomial, Ψ is the basis function (which in this case is the Chebyshev polynomials), $\{a_i\}_{i=0}^{K}$ are unknown coefficients, ω_i are the Chebyshev nodes, and $O(K)$ is an approximation error (which is of order 10^{-15} in the solutions I provide). The Chebyshev nodes are

$$\omega_i = \cos\left(\frac{2i+1}{2(K+1)}\pi\right), \; i = 0, ..., K.$$

Therefore, $\omega_i \in [-1,1]$. Since in the model $x \in (0,1)$, I express the domain as [21] $x_i = \frac{1}{2}(1+\omega_i)$, and therefore x never reaches 0 or 1 for finite K. The Chebyshev polynomials of order $j > 2$ can be represented in the following recursive form:

$$\begin{aligned} \Psi_0 &= 1 \\ \Psi_1 &= \omega \\ \Psi_{j+1} &= 2\omega\Psi_j - \Psi_{j-1}. \end{aligned} \tag{47}$$

Based on (47), it is straightforward to compute the derivatives of $h(x)$ using (46). The rest of the procedure is to solve for the associated set of unknown coefficients as $\{a_i\}_{i=0}^{K}$ in each function, such that equilibrium conditions are verified. Since the state variable x is strong Markov, based on Duffie and Lions (1992), the founded solution for the value functions is unique. Solving PDEs for a function $h(x,\pi)$ is a direct extension of this logic, by extending the argument to a tensor grid to represent the two-dimensional state space.

[21] For π, which is between π_L and π^{max} —where I π^{max} is set to 5 standard deviations above the mean of π —the nodes are $\pi_i = \pi_L + \frac{\pi^{max}-\pi_L}{2}(1+\omega_i)$.

Invariant distribution of π. Let $g(\pi, t, ; \pi_0)$ be the density process associated with (24). Informally, the invariant distribution is when $\frac{\partial [g(\pi,t;\pi_0)]}{\partial \pi} = 0$ (i.e., g does not depend on time). The Kolmogorov Forward Equation for the process π with initial condition $\pi_0 > \pi_L$ is

$$0 = -\frac{\partial [g(\pi)\mu(\pi)]}{\partial \pi} + \frac{1}{2}\frac{\partial^2 \left[g(\pi)\sigma(\pi)^2\right]}{\partial \pi^2},$$

where $\mu(\pi) = \lambda_\pi(\overline{\pi} - \pi_t)$ and $\sigma(\pi) = \sigma_\pi\sqrt{\pi_t - \pi_L}$, as in (24). So, $0 = \frac{\partial}{\partial \pi}\left\{-g(t,\pi)\mu(\pi) + \frac{1}{2}\frac{\partial\left[g(t,\pi)\sigma(\pi)^2\right]}{\partial \pi}\right\}$. I can omit the constant for now, it will be used to integrate the density to one. Changing variables $\widetilde{g}(\pi) = g(\pi)\sigma(\pi)^2$. Then,

$$\frac{\frac{\partial}{\partial \pi}\widetilde{g}(\pi)}{\widetilde{g}(\pi)} = 2\frac{\mu(\pi)}{\sigma(\pi)^2},$$

which means

$$g(\pi) = \frac{1}{\sigma(\pi)^2}\exp\left(2\int^u \frac{\mu(u)}{\sigma(u)^2}du\right).$$

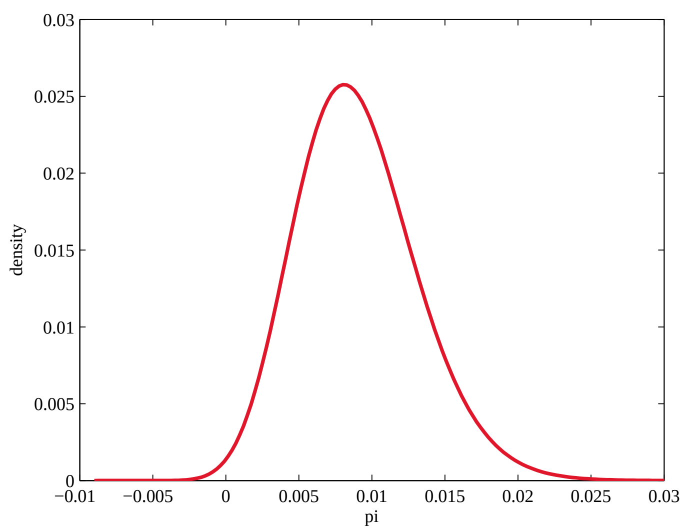
The integral boils down to

$$\int^u \frac{\mu(u)}{\sigma(u)^2}du = \frac{\lambda_\pi\overline{\pi}}{\sigma_\pi^2}\int^u \frac{du}{(u-\pi_L)} - \frac{\lambda_\pi}{\sigma_\pi^2}\int^u \frac{u}{u-\pi_L}du$$

so

$$g(\pi) = \frac{\overline{g}}{\sigma_\pi^2}(\pi - \pi_L)^{2\left(\frac{\lambda_\pi(\overline{\pi}-\pi_L)}{\sigma_\pi^2}\right)-1}\exp\left(-\frac{2\lambda_\pi}{\sigma_\pi^2}\pi\right), \pi \in (\pi_L, \infty] \tag{48}$$

where $\overline{g}$ is a constant to integrate to one. The following is a picture of g under the calibration in the main text.



■