Competing to Coordinate: Crowding Out in Coordination Games *

Sarita Bunsupha † and Saran Ahuja ‡

December 2, 2018

Job Market Paper

Click here for most recent version

Abstract

This paper develops a framework for coordination games that accounts for the role of competition. When crowding out is severe, agents no longer find it optimal to coordinate when everyone else participates. Our setup allows for the characterization of equilibrium outcomes when payoffs are subject to any degree of crowding out and presents a set of comparative statics with respect to the substitutability feature. The model highlights the impact of competition in coordination games, wherein substitutability lowers individual payoffs from coordinating. In many common global game contexts, accounting for the crowding out of payoffs changes widely held intuitions on strategies and on policy implications. For example, in the context of speculative currency attacks, selling a currency after receiving a sufficiently bad signal about reserves may no longer be a dominant strategy; in the presence of substitutability, setting a quota on how many speculators can attack may increase the chance of abandoning the peg; currencies with potentially small depreciation but ample liquidity can be subject to more pressure than currencies with potentially large depreciation but low liquidity.

JEL Classification: C72, D82, D84, F31
Keywords: incomplete information, global games, coordination games, strategic substitutes, speculative currency attacks

*We are grateful to Muhamet Yildiz for his continuous support on this project. Sarita Bunsupha thanks Robert Barro, Emmanuel Farhi, Gita Gopinath, Matteo Maggiori, and Ludwig Straub for invaluable guidance and thanks anonymous referees, and seminar participants at Harvard Finance Lunch, Harvard Games and Markets Lunch, Harvard International Lunch, Harvard Macro Lunch, and MIT Theory Lunch for helpful comments.

†*Corresponding author* sbunsupha@fas.harvard.edu. Department of Economics, Harvard University, Cambridge, MA 02138. `https://scholar.harvard.edu/sbunsupha`

‡saran.ahuja@scbabacus.com. SCB Abacus, Bangkok, Thailand. The participation on this work was during the visit at the Department of Economics, Harvard University in Summer 2017.

1 Introduction

Many real-world phenomena can be characterized as coordination games. Diamond and Dybvig (1983) formalize how coordinated deposit withdrawals can trigger bank runs. Murphy et al. (1989) explain how industrialization occurs only when firms coordinate to bump up aggregate demands. Obstfeld (1996) shows how currency crises originate from self-fulfilling beliefs. Farhi and Tirole (2012) highlight banks' incentives in taking correlated risk.

While these stylized models generate meaningful economic implications, they often omit one key feature. Individual payoffs in those models, however, are independent of the aggregate coordinating size.[1] In practice, there is likely strategic substitutability between agents, as a larger number of people in the coalition lowers individual net returns from coordinating.

The coexistence of strategic complementarity and strategic substitutability prompts a natural question: how do these two forces interact? This paper introduces a substitutability feature into coordination games. In particular, we extend standard static[2] regime-change games to allow payoffs to depend on the aggregate attacking size.

The paper focuses on a speculative currency attack as a main application. A country abandons its currency peg in the next period if it faces sufficiently high selling pressure in the current period. Investors who previously sold at the pegged level realize their profits by buying back home currencies. Depending on the market's ability to absorb demand imbalances, this external buying flow can have a non-negligible effect on the floating level.

In the complete information, investors observe the central bank's reserve threshold directly. A sufficiently strong level of reserve will make it dominant for speculators not to attack, and the fixed exchange regime survives. On the hand, when the observed fundamental is weak, investors find it dominant to attack only when individual payoffs remain positive when everyone attacks. Otherwise, investors will attack with a strictly mixed strategy that yields zero individual returns. The resulting regime when the fundamental is in this region is the floating one. When the fundamental is nether too weak nor too strong, investors can coordinate to either do not attack or attack with the aforementioned positive probability. There are two possible outcomes of exchange regimes in this so-called crisis region.

[1]This is true in most models with incomplete information such as Morris and Shin (1998) for currency crises, Rochet and Vives (2004) for bank runs, Morris and Shin (2004a) for debt crises, and Edmond (2013) for political riots.

[2]It would be interesting to extend the analysis in this paper to look at the impact of substitutability in dynamic global game setups such as those in Angeletos et al. (2007) and He and Xiong (2012).

When investors observe a common noisy signal about the fundamental, they face not only strategic uncertainty but also fundamental uncertainty. Investors now have to worry whether the aggregate attacking size is enough to overthrow the current regime, and any mixed strategy yielding an expected payoff of zero can be sustained as an equilibrium strategy.

In both the complete information and the public signal cases, common knowledge aids coordination, resulting in multiple equilibria as in standard coordination games.

Global games perturbation outside the supermodular setting remains an open question. This paper offers a leading example of how to solve such a problem by characterizing an equilibrium strategy when investors observe private signals with uniform noise.

The payoffs are said to have a weak substitutability if individual payoffs remain non-negative when everyone launches an attack. In this case, both the left and the right dominance regions exist. Attacking dominates when observing extremely low private signals, while no attack dominates for exceedingly high private signals. In this case, there is a unique monotone pure-strategy Nash equilibrium.

The equilibrium characterization becomes significantly more complicated when the substitutability is strong: individual payoffs turn negative if everyone attacks. When investors see sufficiently weak signals, they are certain that any attack will be successful. If no one attacks, they have an incentive to deviate and attack. On the other hand, if everyone attacks, individual payoffs become negative due to severe crowding out. The left dominance region vanishes. We show that neither finite-switching pure strategies nor monotone strategies can be sustained in an equilibrium.

When there is strong substitutability, investors' strategies no longer admit natural orderings. Milgrom and Shannon (1994)'s single crossing property is no longer satisfied, and Athey (2001)'s generalization to guarantee an existence of Nash equilibria does not apply. Nonetheless, we can give a full characterization of a pure-strategy equilibrium, proving a non-trivial existence result.

When the substitutability is strong, we show that the constructed infinitely-switching pure strategy is unique under a class of strategies with an aggregate action that is monotone in the fundamental. As private signals become precise, the convergent strategy is a monotone threshold strategy. Investors do not attack if they observe signals above a specified threshold and attack with a certain positive probability otherwise. Both the switching threshold and the specified attacking probability are monotone and continuous with respect to the substitutability. Therefore, our results can be viewed as a natural extension of standard regime-change games.

The paper provides a complete equilibrium characterization for any given payoff function and level of noise. We are then equipped to study the interplay between strategic complementarity and strategic substitutability.

When we can rank the level of substitutability, we show that equilibrium outcomes are continuous and monotone with respect to the substitutability feature. The more crowding out, the lower the individual payoffs from attacking. For a given signal, investors are less likely to attack. As a result, the fundamental switching threshold moves to the left, and the regime survives for a wider range of the fundamental.

Even when payoff functions are not rankable, we can still analyze the trade-off between returns and liquidity. We show that both the level and the rate of substitutability matter. It is no longer obvious which currency is more likely to be overthrown: the one with a larger potential depreciation but with a higher price impact of trading size or the one with lower potential returns but higher liquidity. Our closed-form characterization enables us to answer such question precisely.

We then revisit the impact of external selling pressure and policies such as a quota on the probability of regime switching. Without substitutability, external attacking pressure aids coordination without compromising individual net returns. Such external attacking pressure always increases the chance of regime switching. With crowding out, this external selling pressure has an additional effect of lowering individual payoffs. Our analysis reveals that the coordination motive always dominates. External selling pressure will increase the chance of regime floating regardless of the severity of crowding out.[3]

Similarly, we can reevaluate the impact of a selling quota. In the standard regime-change game, a quota deters the sale of currencies, as it limits the coordination size. In the presence of substitutability, a quota may alleviate the crowding out, increasing an incentive to sell. We show that the impact of a quota depends on the severity of substitutability and the level of the imposed quota.

In addition to the speculative currency attacks framework, our model can be generalized further. Results follow for any regime-switching games with the benefit of a successful attack that is monotonically decreasing with the size of the aggregate attack and/or the attacking cost that is monotonically increasing with the aggregate attacking size. We discuss how the setup can be applied for other applications such as venture capital investment, debt rollover, and overthrowing a dictator in the paper.

While global games, first studied by Carlsson and Van Damme (1993) and advanced

[3]The analysis of the impact of external selling pressure is a simplified study of the effect of a big player in the market. In Corsetti et al. (2004), Corsetti et al. (2002), and Davila and Walther (2017), a big player has his private information set and needs to decide whether to launch a binary action or not. Our analysis of external selling pressure assumes that this big player always launches an attack. As an extension, we can study how our analysis changes when incorporating a decision made by a big player.

and summarized by Morris and Shin (2003), emerged as an equilibrium selector for games with strategic complementarity, the uniqueness of more generalized games cannot be easily established. Morris and Shin (2009) demonstrate that uniqueness fails commonly in games with strategic substitutability. Such difficulty results in few papers that analyze games with both strategic complementarity and strategic substitutability.

Our paper is not the earliest work featuring both strategic complementarity and strategic substitutability but is the first to successfully specify equilibrium outcomes. Karp et al. (2007), henceforth KLM, study equilibrium properties of incomplete information games where an aggregate attacking size and a strength of the fundamental enter payoff functions separately and additively. They prove that a distributional-strategy equilibrium exists without giving a proper characterization.[4]

Embedding substitutability into regime-change games make economic models more realistic. Goldstein and Pauzner (2005) point out that depositors' incentive to withdraw early is highest when the number of withdrawals is just enough to trigger the bank run. Their setup is different form ours. Both the left and the right dominance regions exist in their model, and the strategic substitutability is weak whenever the fundamental falls into a range that requires coordination.

Recently, He et al. (2015), henceforth HKM, note that sovereign debt investments have both strategic complementarity and strategic substitutability features. A larger number of buyers lowers rollover risk but drives up the bond prices, lowering bond returns. Our paper differs from their setup as follows. Investors in HKM have no choice but to put their money in one of the two bonds, both of which have rollover risks. When the aggregate funding is abundant, substitutability dominates coordination. HKM focus on the effect of debt size on rollover risk versus returns. Our model portrays the trade-off between returns and liquidity. Lastly, HKM focus mainly on applications and abstracts away from the fundamental uncertainty. We consider the standard private information structure with both fundamental and strategic uncertainty.

The paper proceeds as follows. Section 2 features our setup of the speculative currency attacks model. Section 3 discusses equilibria under common knowledge. Section 4 analyzes equilibrium outcomes under private signals. Section 5 discusses the role of crowding out in coordination games and potential policy implications. Section 6 shows the generalized version of our model and discusses other applications to economics. We conclude our findings and contributions in Section 7.

[4] Hoffmann and Sabarwal (2015) later discover the gap in the proof and argue that KLM's arguments apply only in the finite-player setting.

2 Model

This section considers the modified version of speculative currency attacks. There are three types of agents: a central bank, speculators, and international liquidity providers. There are two exchange rate regimes: a fixed exchange regime and a floating exchange regime.

At the beginning, a central bank fixes an exchange rate at $\bar{d}$, normalized to one (dollar per home currency). Speculators buy and sell home currency directly with the central bank at this specified rate. If a central bank decides to abandon the peg, we enter the floating exchange regime. Speculators now buy and sell currencies from international liquidity providers.

There is a continuum of speculative traders indexed by $i \in [0,1]$. In period 0, each trader chooses whether to short sell the home currency $(a_i = 1)$ or not $(a_i = 0)$. There is a transaction cost $t > 0$ associated with short selling. The fraction of traders choosing to sell is denoted by $A = \int_0^1 a_i di$.

A central bank derives a private benefit of $V > 0$ if the peg is maintained. The cost of maintaining the peg, denoted by $N(\cdot)$, is strictly increasing in the net sell order O, i.e. $N'(O) > 0$. The threshold where the benefit and the cost are just balanced out is called the fundamental θ such that $V = N(\theta)$. This θ governs the strength of the regime. A central bank abandons the peg in period 1 if and only if the aggregate attacking size is no less than the regime's strength, i.e. $A \geq \theta$. Once the peg is abandoned, international liquidity providers absorb investors' demand imbalances at the floating exchange rate of $d(A) \geq 0$.

The speculator's payoff under action a_i when the aggregate attacking size is A and the fundamental is θ, denoted by $u(a_i, A, \theta)$, can be written as:

$$u(a_i, A, \theta) = a_i((1 - d(A))\mathbf{1}_{A \geq \theta} - t), \tag{1}$$

where $d(\cdot), t$, and θ are model parameters. Traders who previously sold at 1 need to buy back at $d(A)$ to capture their profits. The buying demand in period 1 can potentially affect the floating level, that is d is allowed to be a function of A.

Table 1 summarizes individual payoffs. Figure 1 illustrates an individual attacking payoff as a function of the aggregate attacking size A for a given fundamental θ.

Order Flow and Flexible Exchange Rates

The existing literature on regime-change models often assumes constant individual payoffs when the attack is successful irrespective of the size of the aggregate attack. In that case, $d(A) \equiv \bar{d}$ for some constant $d \in \mathbb{R}_+$. However, international liquidity providers might face some frictions and may charge higher prices if they need to provide bigger

	Floating ($A \geq \theta$)	Fixed ($A < \theta$)
Sell ($a_i = 1$)	$1 - d(A) - t$	$-t$
No sale ($a_i = 0$)	0	0

Table 1: Individual Payoffs.

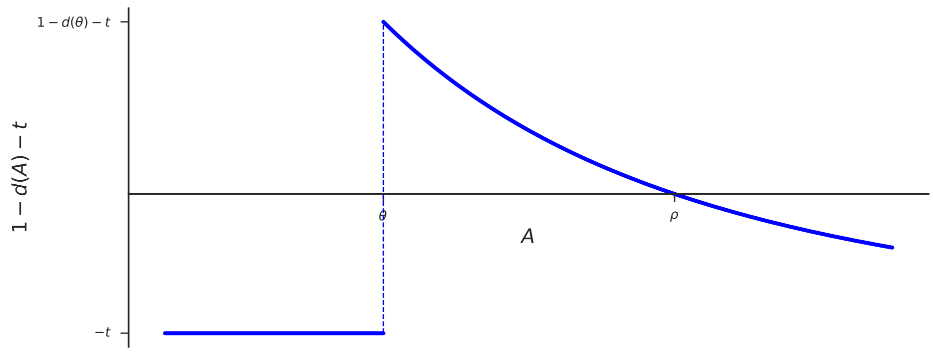


Figure 1: Individual attacking payoffs as a function of aggregate attacking size A

liquidity. We assume $d(A)$ to be non-decreasing in A.

Assumption 1. *$d(\cdot)$ is continuous and monotonically increasing with $1 - d(0) > t$.*

Assumption 1 assumes the continuity of the floating exchange level for the simplicity of the proof. $1 - d(0) > t$ ensures that without the effect of crowding out, the potential depreciation is big enough that profits from selling can cover the short-selling cost.

Two possible explanations of why the floating exchange rate $d(A)$ might be increasing in the aggregate attacking size A are as follows.

1. **Gabaix and Maggiori (2015)**

   International liquidity providers absorb portfolio imbalances but have limited risk-bearing capacities in doing so. We take the buying demand of home currencies in period 1 to be an external demand flow. In this case, we have that

   $$d(A) = \tilde{d} + \frac{\Gamma A}{2 + \Gamma},$$

   where $\tilde{d}$ is the fundamental floating level governed by demands of domestic and foreign goods, and $\Gamma \geq 0$ captures the degree of market imperfection.

2. **Convex Cost in Providing Liquidity**

   Let $\tilde{d}$ be the fundamental floating exchange level. International liquidity providers choose a number of units of the currency that they will provide, while facing a convex cost of providing liquidity. Such cost is denoted by $L(A)$ with $L'(A) > 0$ and $L^{''}(A) > 0$. Let $d(A)$ denote the equilibrium per-unit price of the home currency. The international liquidity provider faces the profit maximization problem as below:

$$\max_{A} d(A)A - (\tilde{d}A + L(A)).$$

The optimality condition requires that $d(A) = \tilde{d} + L'(A)$. The equilibrium exchange rate will be increasing in the aggregate attacking size A whenever the liquidity cost $L(A)$ is convex. For example, if $L(A)$ has the following functional form: $L(A) = \gamma\frac{A^{1+\phi}}{1+\phi}$ with $\gamma \geq 0$ and $\phi \geq 0$, then the effective period-1 floating exchange level is $d(A) = \tilde{d} + \gamma A^{\phi}$.

Degree of Substitutability

When attacking remains profitable when everyone else attacks and the attack is successful, i.e. $1 - d(1) > t$, the substitutability is weak. Formally, define $\rho \in (0, 1]$ as follows.

$$\rho = \sup\{A \in [0, 1]; 1 - d(A) \geq t\}. \tag{2}$$

The substitutability is weak if and only if $\rho = 1$. When $\rho < 1$, crowding out makes individual rewards no longer cover the attacking cost when everyone attacks, i.e. $1 - d(1) < t$. ρ gives some information about the degree of substitutability, but we shall see that the entire characterization of $d(\cdot)$ matters for the equilibrium characterization.

3 Equilibrium under Common Knowledge

We first analyze outcomes under common knowledge, as the limiting private information case will pick a unique strategy out of multiple equilibrium strategies in the common knowledge case.

3.1 Complete Information

There is no fundamental uncertainty when each speculator observes the underlying fundamental θ. We say that an investor adopts a strategy $p \in [0, 1]$ whenever his equilibrium strategy is selling with probability p. Figure 2 summarizes strategies over the range of all possible fundamentals. A formation characterization of the equilibrium set is stated in Appendix A.

When the fundamental is higher than 1, any coordinated attack will never be successful. For fundamental between ρ and 1, i.e. $\theta \in (\rho, 1]$, more than ρ fraction of speculators must launch an attack to successfully overthrow the fixed exchange regime. However, net individual payoffs become negative even when the attack is successful. Therefore, no attack ($a_i = 0$) is a dominant strategy for sufficiently strong fundamentals ($\theta > \rho$). We call this region "the right dominance region".

Without substitutability, there is another dominance region when the fundamental θ is sufficiently low ($\theta \leq 0$). In this region, any size of aggregate attack $A \in [0, 1]$ will lead to a successful attack. Attacking ($a_i = 1$) is strictly dominant. In our setting, attacking remains a dominant strategy whenever the degree of substitutability is weak ($\rho = 1$). When the substitutability is strong ($\rho < 1$), attacking is no longer profitable. The left dominance region vanishes. The only equilibrium sustainable when the fundamental is sufficiently low is attacking with a probability $\rho < 1$, yielding a net payoff of zero.

For a fundamental that is neither too strong nor too weak, i.e. $\theta \in (0, \rho]$, two equilibria coexist. Speculators coordinate either to take no action or to sell with probability ρ. Individual payoffs from both strategies are equal to 0, but the resulting regime will be different. The fixed exchange rate regime remains in the no-attack equilibrium, while the regime switches to floating in the ρ-attacking equilibrium. Multiple equilibria exist in this "crisis region". The two distinct resulting regime captures the idea of self-fulfilling attack.

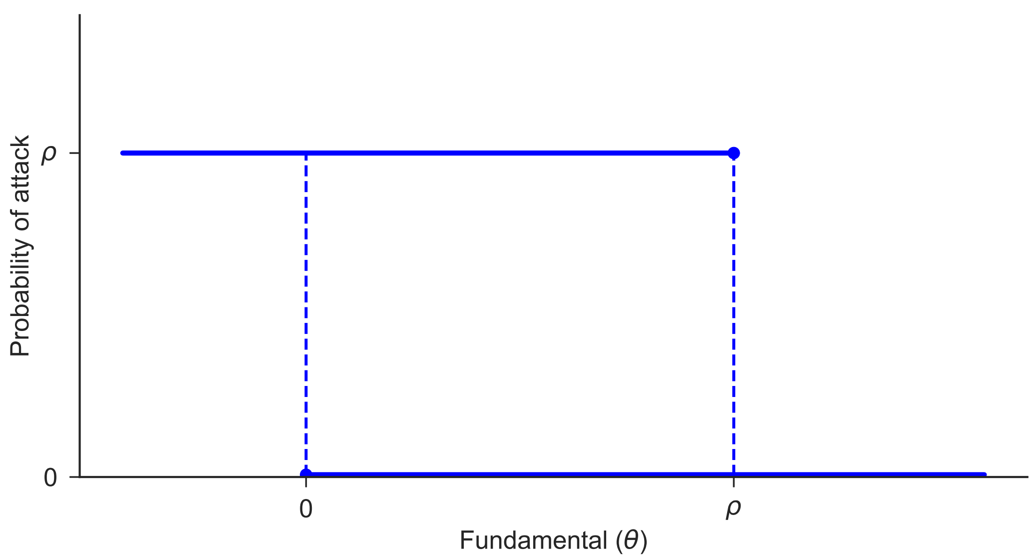


Figure 2: Equilibria for the complete information case

3.2 Public Signal

Information frictions are prevalent in reality. Agents in the economy rarely observe the exact level of fundamentals but instead observe news that is related in a certain way to the true fundamental. This section analyzes equilibrium outcomes when speculators cannot observe fundamental directly. Instead, they observe a public signal y, which is a noisy signal of the fundamental:

$$y = \theta + \epsilon, \quad \epsilon \sim U[-e, e],$$

where ϵ is a uniformly distributed noise. Each trader is equipped with (improper) uniform prior on the real line so that the posterior distribution of θ is uniformly distributed with mean y and the same variance as that of the noise.

Remark 2. *Assumptions of a uniform prior and a uniform posterior enable us to obtain an explicit characterization of equilibrium outcomes. Results in this section extend to more general noise structures without loss of generality.*

Without loss of generality, we restrict our attention to symmetric Nash equilibria, where each speculator takes the same action[5].

A symmetric strategy that each speculator sells with probability $p \in [0, 1]$ leads to an aggregate attacking size p. Given that such strategy is played in an equilibrium, the expected payoff from attacking conditional on observing a public signal y is given by

$$\mathbb{E}[u(1, p, \theta) \mid y] = (1 - d(p))\mathbb{P}(\theta \leq p \mid y) - t, \tag{3}$$

where $\mathbb{P}(\cdot \mid y)$ denotes a conditional probability given a public signal y.

No attack is an equilibrium strategy if and only if the expected payoff from selling when no one else attacks is no greater than the expected payoff from not selling. Equivalently, $\mathbb{E}[u(1, 0, \theta) \mid y] \leq 0$. Similarly, attacking is an equilibrium strategy if and only if the expected payoff from selling when everyone else attacks is no less than the expected payoff from no attack. That is, $\mathbb{E}[u(1, 1, \theta) \mid y] \geq 0$. A speculator will play a strictly mixed strategy $p \in (0, 1)$ if and only if he is indifferent between attacking and no attack when the aggregate attacking size is p, i.e. when the following indifference condition holds

$$\mathbb{E}[u(1, p, \theta) \mid y] = 0. \tag{4}$$

For arbitrary payoff functions $(d(\cdot), t)$, a private signal y, and a noise $e > 0$, a mixed

[5] Suppose equilibrium strategies are asymmetric. As all speculators receive the same expected payoff, they must be indifferent among these asymmetric strategies. Therefore, we can replace any asymmetric strategies with an equivalent symmetric mixed strategy.

strategy $p \in (0,1)$ can sustain a Nash equilibrium if the indifference condition (4) holds. That is,

$$F_\theta(p \mid y) := \mathbb{P}(\theta \leq p \mid y) = \frac{t}{1-d(p)}. \tag{5}$$

When the noise e is large, there might be an arbitrary number of solutions $p \in (0,1)$ for a given private signal y. However, when the noise goes to zero, most of these equilibria vanish. Figure 3 provides an example of plots of $f(p) = \mathbb{P}(\theta \leq p \mid y)$ and $g(p) = \frac{t}{1-d(p)}$ for a given y. Intersections of f and g yield mixed-strategy equilibrium strategies.

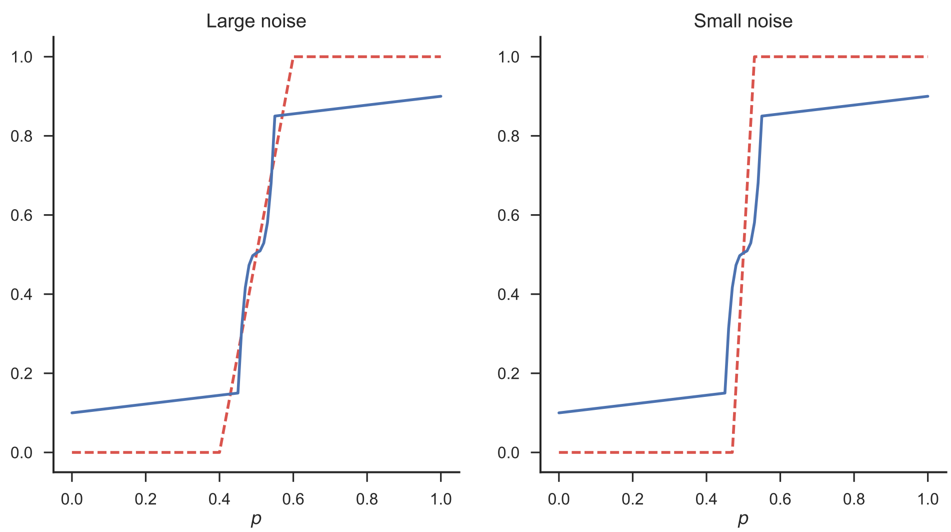


Figure 3: Plot of $f(p) = \mathbb{P}(p \geq \theta \mid y)$ (dash) and $g(p) = \frac{t}{1-d(p)}$ (solid) for p in $(0,1)$ when $y = 0.5$

The right-hand side of Figure 3 illustrates that there is at most one intersection between f and g for p in $(y-e, y+e)$ when the noise is sufficiently small. We denote such point by $\delta(y)$.

Figure 4 illustrates an equilibrium correspondence when the noise is sufficiently small. A formal proposition characterizing the set of equilibrium strategies for such case can be found in Appendix A.

Compared to the complete information case, we identify an extra equilibrium strategy $\delta(y)$. This equilibrium strategy arises from the common uncertainty of the fundamental level. At this mixed-strategy equilibrium, the expected gain from a successful attack is equal to the expected loss due to a failed attempt. The emergence of this extra equilibrium strategy is due to the structure of regime-change models and is not a result of the added substitutability feature.

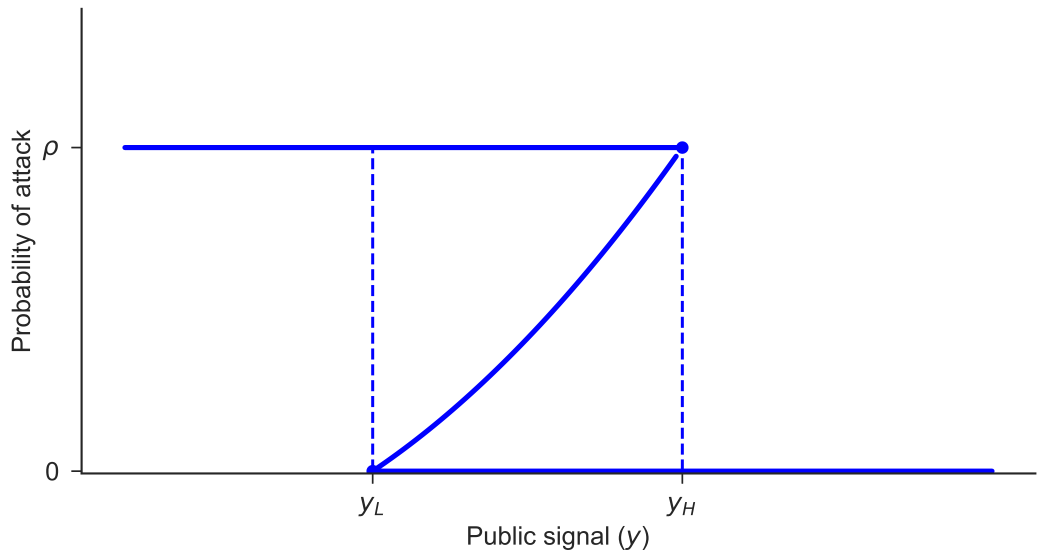


Figure 4: Equilibria for the public signal case when noise is sufficiently small

4 Equilibrium with Private Signals

Common knowledge is a restrictive assumption that aids the equilibrium analysis substantially. In real life applications, speculators are exposed to different sets of information and may use distinctive technologies in interpreting the data. This section analyzes the setup in which each speculator observes his own private signals.

Formally, each speculator receives a private signal x_i:

$$x_i = \theta + \epsilon_i, \quad i \in [0,1],$$

where $(\epsilon_i)_{i\in[0,1]}$ is independently uniformly distributed over $[-e, e]$. The posterior distribution of θ given a signal x_i is assumed to be uniform over $[x_i - e, x_i + e]$.

With individual-specific signals, speculators form beliefs on both the fundamental and the distribution of signals observed by the remaining speculators. The posterior distribution of other speculators' signals conditional on observing a private signal x_i is as follows:

$$x_j = \theta + \epsilon_j = x_i - \epsilon_i + \epsilon_j.$$

In the uniform noise case, this posterior distribution spans over an interval $[x_i - 2e, x_i + 2e]$, as displayed in Figure 5.

Again, we look at the symmetric-strategy equilibria, i.e. equilibria in which speculators with the same private signal take the same action.

A measurable function $s : \mathbb{R} \to [0,1]$ denotes speculators' strategy profile. $s(x_i)$ in-

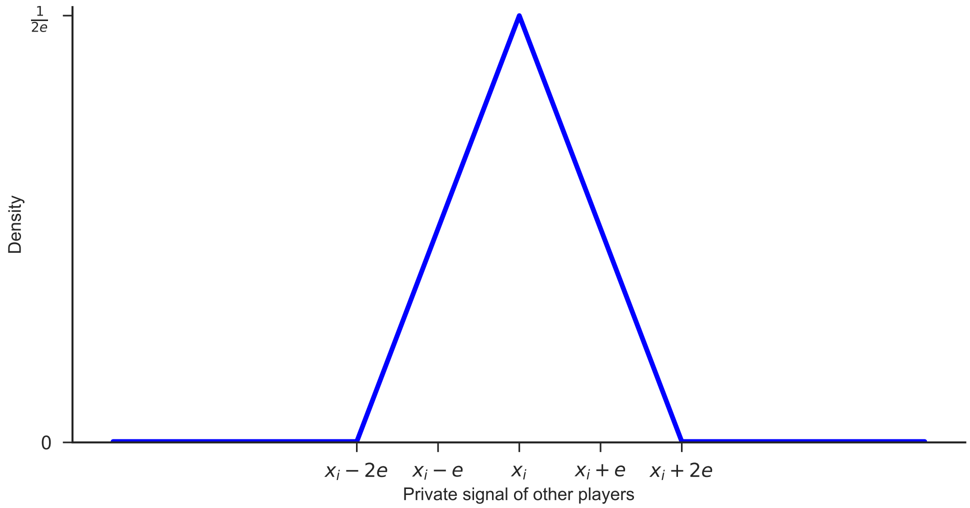


Figure 5: Posterior distribution of other players' signal x_j given a private signal x_i

dicates a probability of selling conditional on observing a private signal x_i. We define a corresponding *aggregate action* $A^s : \mathbb{R} \to [0,1]$ by

$$A^s(\theta) = \frac{1}{2e} \int_{\theta-e}^{\theta+e} s(u)du. \tag{6}$$

If speculators follow a strategy profile s, the aggregate action $A^s(\theta)$ represents a fraction of speculators attacking the currency when the actual realized fundamental is θ. Since this fundamental θ is unobservable, the aggregate action $A^s(\theta)$ is also unobservable.

Given a strategy profile s, an expected payoff from attacking of a speculator with a private signal x_i, denoted by $F^s(x_i)$, can be written as:

$$\begin{aligned}
F^s(x_i) &= \mathbb{E}[u(1, A^s(\theta), \theta)|x_i] \\
&= \int_{\mathbb{R}} \left((1 - d(A^s(\theta)))\mathbf{1}_{A^s(\theta) \geq \theta} - t \right) f(\theta|x_i) d\theta \\
&= \frac{1}{2e} \int_{x_i-e}^{x_i+e} \left(1 - d(A^s(\theta)))\mathbf{1}_{A^s(\theta) \geq \theta} - t \right) d\theta \\
&= \frac{1}{2e} \int_{x_i-e}^{x_i+e} \left(1 - d(A^s(\theta)))\mathbf{1}_{A^s(\theta) \geq \theta} \right) d\theta - t,
\end{aligned} \tag{7}$$

when $f(\theta|x_i)$ denotes the posterior density of the fundamental θ conditional on observing a private signal x_i. We will omit the superscript s in A^s when it creates no confusion in the context.

4.1 Strong Substitutability: $\rho < 1$

This section first proves by construction the existence of a pure-strategy equilibrium and shows that the established equilibrium is unique under a class of equilibria with the aggregate attacking size that is monotone in the fundamental. Moreover, the constructed equilibrium converges to a monotone mixed-strategy equilibrium as noise vanishes.

Existence

We first restrict our search to pure-strategy equilibria. Sufficiently strong private signals deter any attack, making no attack dominant. The right dominance region exists. Observing sufficiently weak private signals guarantee that any size of an aggregate attack can trigger a regime switching. However, it is no longer optimal to attack if everyone does so. Too big aggregate attacking size crowds out individual returns. The following lemma states that the range of signal intervals where speculators take the same pure-strategy action cannot be too large compared to the signal's noise.

Lemma 3 (Infinite-Switching Pure-Strategy)**.** *With a strong substitutability, i.e. $\rho < 1$, consider any pure-strategy Nash equilibrium profile $s : \mathbb{R} \to \{0, 1\}$. If s is constant over any interval $I = (\underline{x}, \bar{x})$ with an upper bound $\bar{x} \leq 0$, then the length of this interval cannot be too large compared to the size of signal's noise, i.e. $|I| = \bar{x} - \underline{x} < 4e$.*

Proof. We prove by contradiction. Suppose there exists a pure-strategy Nash equilibrium profile s and an interval $I = (\underline{x}, \bar{x})$ with $\bar{x} \leq 0$ and $|I| \geq 4e$. Let $x_m = \frac{\underline{x}+\bar{x}}{2}$. If $s(x) \equiv 0$ on I, then $A(\theta) = 0 \geq \theta$ for all $\theta \in [x_m - e, x_m + e]$, and $F^s(x_m) = 1 - d(0) - t > 0$. Attacking yields strictly higher payoffs than no attack, contradicting the fact that s is an equilibrium strategy. Similarly, if $s(x) \equiv 1$ on I, then $A(\theta) = 1 \geq \theta$ for all $\theta \in [x_m - e, x_m + e]$. Therefore, $F^s(x_m) = 1 - d(1) - t < 0$. Deviating to no attack yields a strictly higher payoff, again contradicting the fact that s is an equilibrium strategy. □

Lemma 3 is similar to Proposition 3 in Karp et al. (2007). For any pure-strategy Nash equilibrium, the length of signal intervals where the strategy is constant must be small compared to the variance of the noise. As a signal space spans the real line, this lemma implies that any pure-strategy equilibrium profile must switch between attacking and no-attack infinitely many times.

We next state an existence result by giving the exact characterization of an infinitely-switching pure strategy that can sustain a Nash equilibrium.

Theorem 4 (Existence of Pure-Strategy Equilibrium)**.** *The following pure-strategy profile de-*

scribes a Nash equilibrium of regime-change games when the substitutability is strong, i.e. $\rho < 1$.

$$s^e(x_i) = \begin{cases} 0 & ; x_i \geq \bar{x}^e \\ 1 & ; \bar{x}^e - 2ek - 2e\rho \leq x_i < \bar{x}^e - 2ek, \quad k = 0, 1, 2, \ldots \\ 0 & ; \bar{x}^e - 2e(k+1) \leq x_i < \bar{x}^e - 2ek - 2e\rho, \quad k = 0, 1, 2, \ldots \end{cases}, \tag{8}$$

when $\bar{x}^e$ *is given by*

$$\bar{x}^e = -e + (1 + 2e)(\rho - \bar{u}), \tag{9}$$

and $\bar{u}$ *is the unique value* $u \in (0, \rho)$ *satisfying*

$$\int_{\rho - u}^{\rho} (1 - d(A) - t) dA = (\rho - u)t.$$

Proof. See Appendix B $\square$

The top image of Figure 6 illustrates the strategy profile from Theorem 4. There is a cutoff $\bar{x}^e$ above which speculators never attack. To the left of this cutoff, attacking intervals with a length of $2e\rho$ alternate with no-attack intervals with a length of $2e(1 - \rho)$ indefinitely.

This equilibrium strategy is clearly non-monotonic in private signals. Nonetheless, the aggregate attacking size associated with the constructed equilibrium strategy is smooth and monotone in the (unobserved) fundamentals, as displayed in the bottom image of Figure 6 .

Given the aggregate attacking size A^s, each speculator calculates an expected reward from attacking using equation (7). The top image of Figure 7 displays the aggregate attacking size along with the fundamental threshold $\hat{\theta}$ below which the regime switches. Note that this fundamental switching threshold $\hat{\theta}$ is independent of the noise and always equals to $\rho - \bar{u}$.

Knowing the aggregate attacking size, speculators can calculate an expected payoff from attacking using equation (7). The bottom image of Figure 7 illustrates an expected attacking payoff when all speculators follow the strategy s^e. When this expected payoff is positive, attacking is dominant. No attack is dominant whenever the expected payoff is negative. Speculators are indifferent whenever the expected payoff is zero. In an equilibrium, the expected payoff $F^s(x_i)$ must rationalize the strategy $s^e(x_i)$.

For sufficiently high private signals, i.e. $x_i \geq \hat{\theta} + e$, the attack is never successful, and it is dominant not to attack. As speculators observe slightly lower private signals, the chance of successful attack becomes positive. The expected payoff from attacking strictly

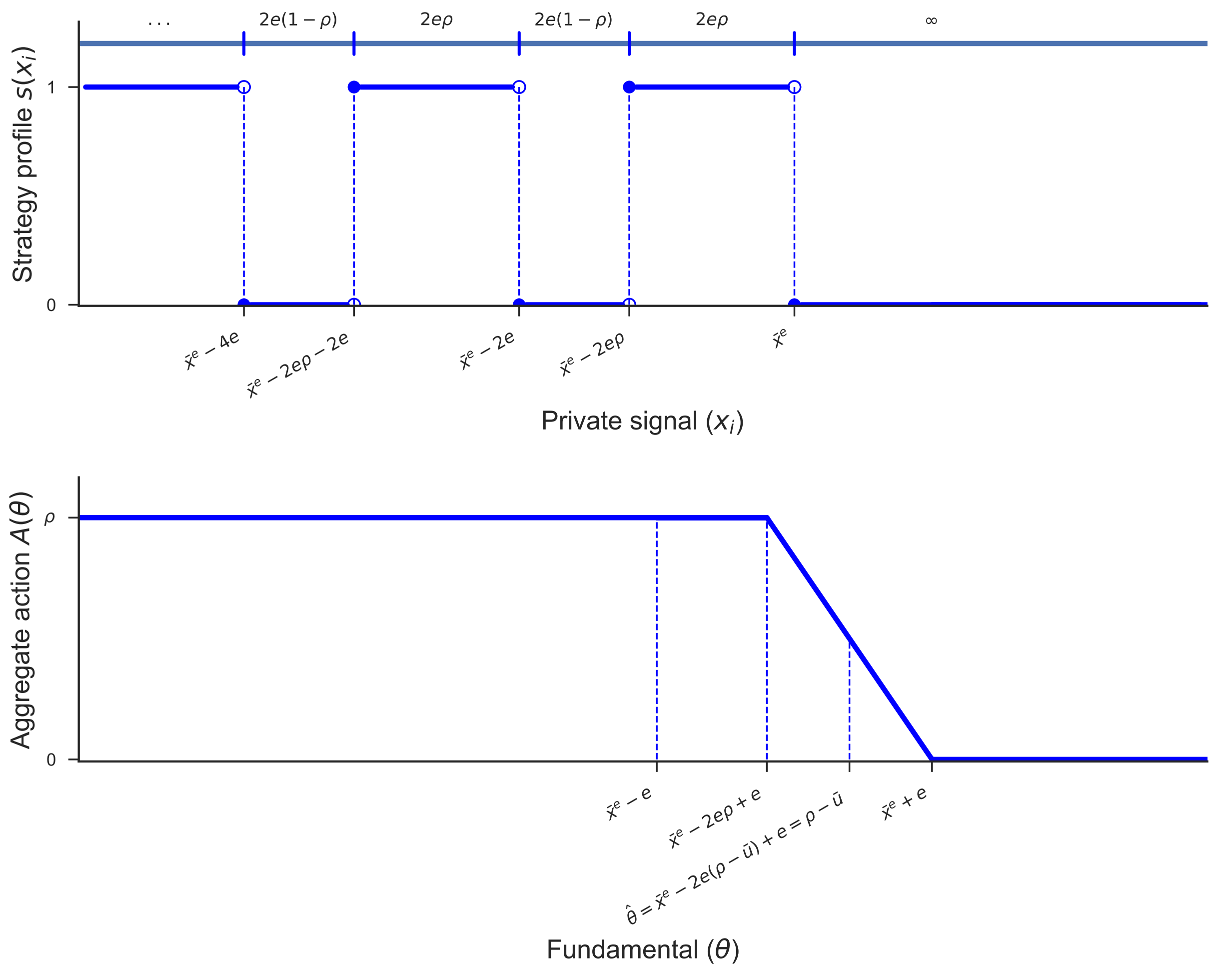


Figure 6: Strategy profile $s(x_i)$ as a function of private signal x_i with internal lengths on top (top). Aggregate action $A(\theta)$ as a function of unobserved fundamental θ (bottom).

increases, as the private signal moves to the left. When speculators observe a private signal $x_i = \bar{x}^e \in [\hat{\theta} - e, \hat{\theta} + e]$, positive payoffs from a successful attack are just enough to compensate for the foregone cost when the regime persists. Speculators will be completely indifferent.

When a speculator observes a private signal $x_i \in [\hat{\theta} - e, \bar{x}^e)$ instead of $x_i = \bar{x}^e$, a chance of a failed attack is being replaced by a successful attack with an aggregate attacking size that is not too large. Therefore, the expected attacking payoff is strictly positive, and attacking dominates in this region. A speculator with a private signal $x_i \in (\hat{\theta} - 2e\rho, \bar{x}^e)$ knows with certainty that an attack will be successful while enjoying higher rewards since less speculator sells. Attacking remains dominant in this region.

A speculator with a private signals $x_i < \bar{x}^e - 2e\rho$ knows that (1) an attack is always successful, and (2) exactly ρ fraction of speculators sell. The expected payoff from attacking

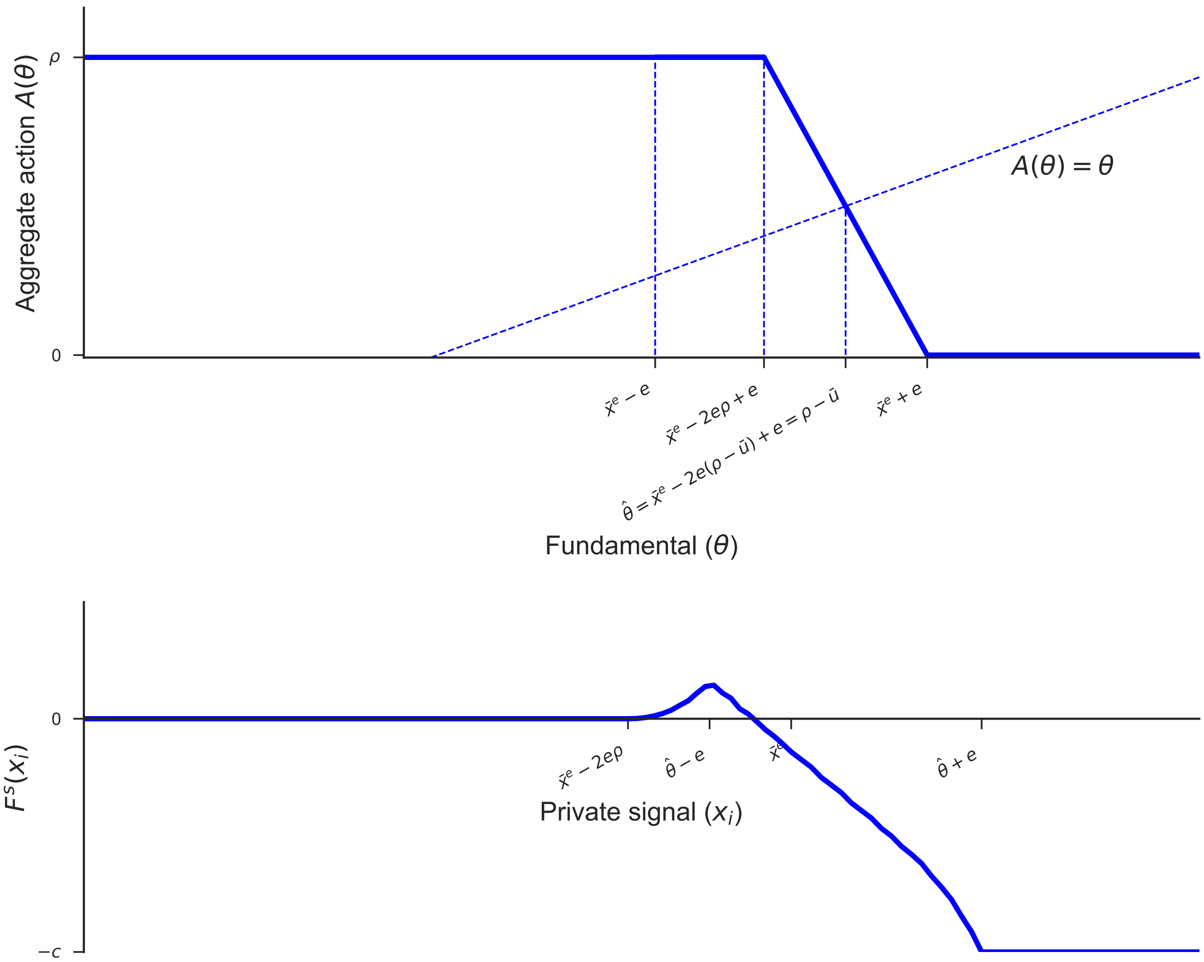


Figure 7: Aggregate action $A(\theta)$ as a function of unobserved fundamental θ (Top). An expected payoff from attacking $F^s(x_i)$ when observing the private signal x_i (bottom).

is exactly zero, leaving speculators indifferent between attacking and no attack.

An existence of a pure-strategy Nash equilibrium when the substitutability is strong is non-trivial. While the right dominance region remains, the left dominance region vanishes. Because of the severe crowding out, neither all attack nor no attack is optimal when receiving exceptionally low private signals. The iterative deletion of strictly dominated strategy used in the standard global games no longer applies here.

In our setting, speculators' strategies do not admit natural orderings. The attacking payoff does not satisfy Milgrom and Shannon (1994)'s single crossing property, and we cannot apply Athey (2001)'s sufficient condition to guarantee an existence of pure-strategy Nash equilibria.

Uniqueness

Though Proposition 3 rules out all monotone pure-strategy equilibria, we have not yet explored an entire universe of non-monotone pure-strategy equilibria as well as mixed-strategy equilibria. Readers may wonder whether our setup has spurious equilibria. The following theorem demonstrates that the constructed equilibrium is unique under a large class of strategy profiles with an aggregate action that is monotone in the fundamental.

First, we define

$$\Psi = \{s : \mathbb{R} \to [0,1]; A^s(\theta) \text{ is monotonic in } \theta\}.$$

The class of strategy profiles Ψ consists of all strategy profiles with an aggregate attacking size that is monotone in the fundamental θ. Strategy profiles in Ψ include not only monotone strategies (pure or mixed) but also a substantial number of non-monotone strategies. The constructed equilibrium profile s^e from equation (8) is non-monotone but falls in a class of Ψ.

Theorem 5 (Uniqueness under Ψ)**.** *For $e > 0$, the strategy profile s^e described in equation* (8) *is a unique equilibrium strategy in Ψ.*

Proof. See Appendix C □

Theorem 5 proves that there is exactly one equilibrium with a monotone aggregate action. Any other equilibrium, if it exists, must have a non-monotonic relationship between the aggregate attacking size and the fundamental.

This (local) uniqueness hints that private information may still work as an equilibrium selector. Common knowledge allows for coordination, which leads to multiple equilibria. Taking away common signals deprives speculators of their coordinating devices and eliminates multiple outcomes.

Convergence

The constructed equilibrium strategy under private signals looks distinct from equilibrium strategies under the common knowledge. However, as the variance of the noise goes to zero, the equilibrium strategy under private signals converges to a monotone mixed-strategy equilibrium and will pick a unique strategy from multiple equilibrium strategies from the common information case.

Figure 8 displays the equilibrium strategy for different levels of noise. As private signals get more precise, the switching between attacking intervals and no-attack intervals becomes more rapid.

Recall that the fundamental switching threshold $\hat{\theta} = \rho - \bar{u}$ is independent of the noise. In the limit of e going to 0, the upper dominant cutoff $\bar{x}^e$ converges to $\bar{x}^0 = \rho - \bar{u} = \hat{\theta}$. That is, the upper dominant cutoff for private signals coincides with the switching threshold for the fundamental.

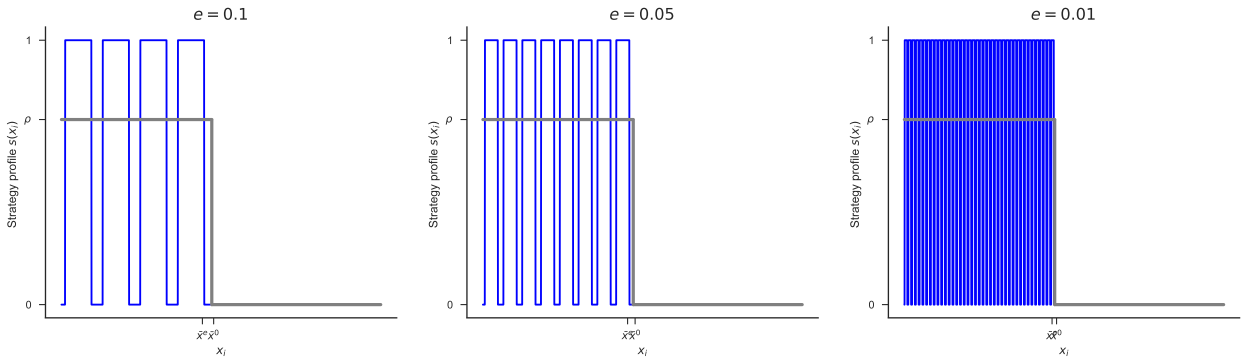


Figure 8: A pure-strategy equilibrium as characterized by equation (8) for different levels of noise

In a sense, the equilibrium strategy converges in distribution to the monotone mixed-strategy equilibrium $s^0(x_i)$ defined as follows:

$$s^0(x_i) = \begin{cases} \rho, & \text{if } x_i \leq \bar{x}^0 \\ 0, & \text{if } x_i > \bar{x}^0 \end{cases}. \tag{10}$$

Figure 8 overlays the equilibrium strategy s^e with a monotone mixed-strategy s^0. We observe that for all sufficiently small noise $e > 0$, a fraction of attacking intervals in any small neighborhood around x_i has a measure of ρ for $x_i < \bar{x}^e$ and 0 for $x_i > \bar{x}^e$.

Proposition 6 states a formal convergence proposition.

Proposition 6 (Convergence to Monotone Equilibrium)**.** *The strategy s^e defined by equation (8) converges in distribution to the strategy s^0, as $e \to 0$. That is, for all $\epsilon > 0$, there exists $\delta > 0$ such that $e \in [0, \delta) \Rightarrow \int_{\theta \in \mathbb{R}} \mid A^{s^e}(\theta) - A^{s^0}(\theta) \mid d\theta < \epsilon$.*

Proof. It is straightforward to check that $A^{s^e}(\theta) = A^{s^0}(\theta) = \rho$ for $\theta < \min(\bar{x}^e + e - 2e\rho, \bar{x}^0)$, $A^{s^e}(\theta) = A^{s^0}(\theta) = 0$ for $\theta > \max(\bar{x}^e + e, \bar{x}^0)$, and $0 \leq A^{s^e}(\theta), A^{s^0}(\theta) \leq \rho$ otherwise. Thus, $\int_{\theta \in \mathbb{R}} \mid A^{s^e}(\theta) - A^{s^0}(\theta) \mid \leq (2e\rho)\rho = 2e\rho^2$, which converges to 0, as $e \to 0$. □

Proposition 6 states that in the limit of precise private signals, a unique equilibrium under Ψ converges to a monotone mixed-strategy equilibrium s^0 defined in equation (10).

The precise private signal case can be thought of as a perturbation of the common knowledge case. Such perturbation uniquely selects an optimal strategy profile out of multiple equilibrium strategies. Precisely, there is a fundamental threshold below which a mixed strategy is selected and above which no attack is played. Figure 9 displays an equilibrium selection by private signals.

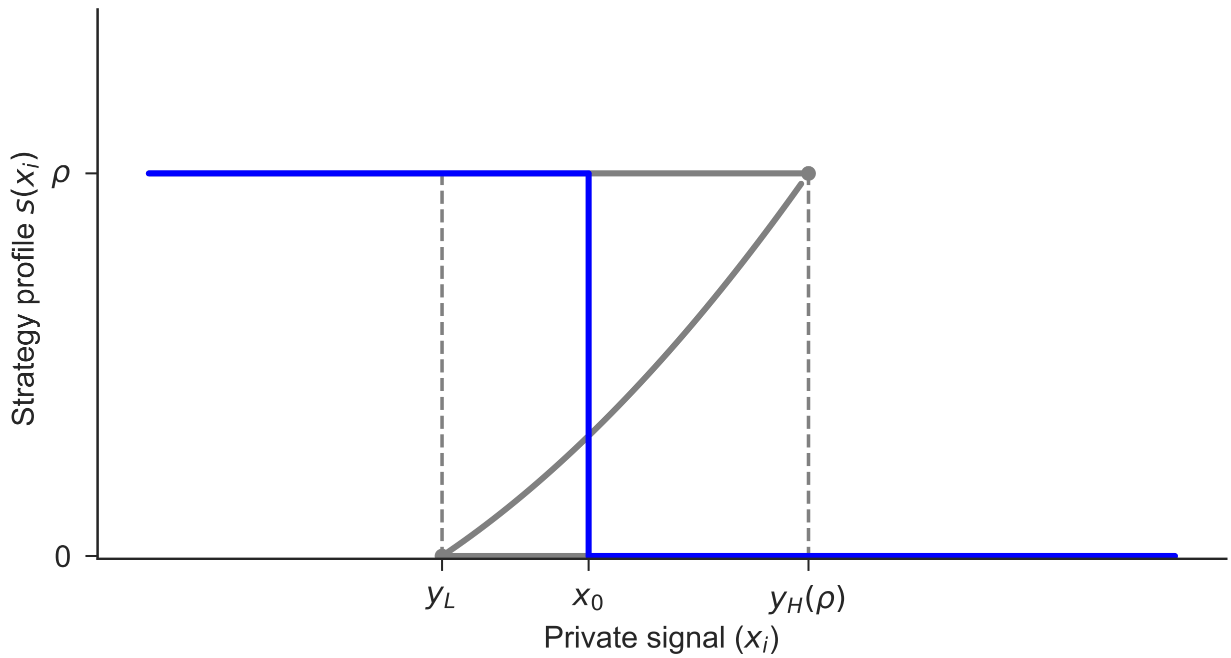


Figure 9: Equilibrium selection by private signals

4.2 Weak substitutability: $\rho = 1$

This section discusses equilibrium outcomes for the case $\rho = 1$ for completeness. With a low degree of substitutability, rewards from attacking remain high enough to cover the attacking cost even when everyone else attacks. The left dominance region reappears. Theorem 7 is parallel to Theoren 4. In this case, the characterized equilibrium is a monotone pure-strategy equilibrium.

Theorem 7 (Existence of Monotone Pure-Strategy Equilibrium)**.** *When $\rho = 1$, the following single switching strategy can be sustained in a Nash equilibrium:*

$$s(x_i) = \begin{cases} 0, & \text{if } x_i > \bar{x}^e \\ 1, & \text{if } x_i \leq \bar{x}^e \end{cases}, \tag{11}$$

where the threshold $\bar{x}^e$ is given by

$$\bar{x}^e = -e + (1 + 2e)(1 - \bar{u}), \tag{12}$$

and $\bar{u}$ is a unique value $u \in (0, 1)$ satisfying

$$\int_{1-u}^{1} (1 - d(A) - t) dA = (1 - u)t.$$

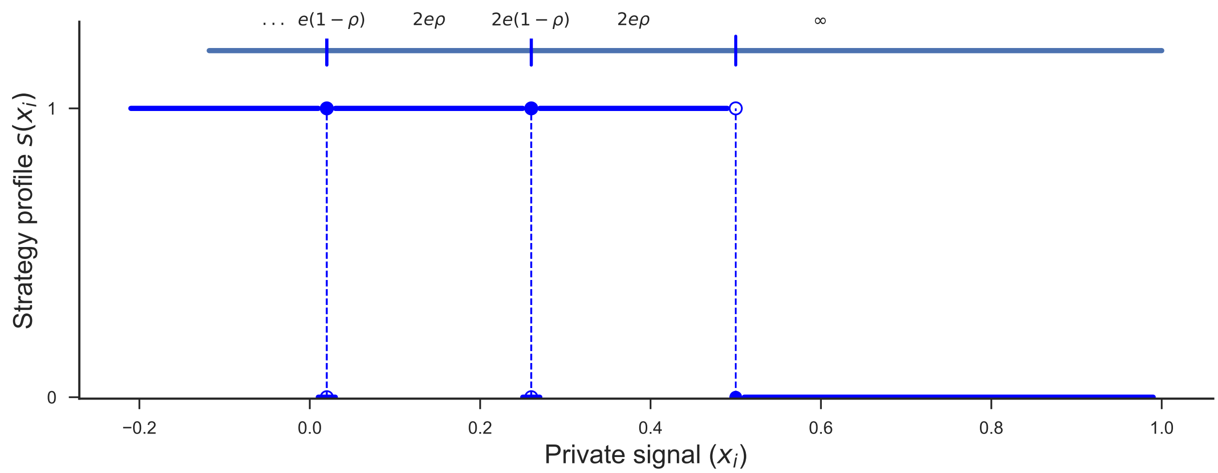


Figure 10: Strategy profile $s(x_i)$ as a function of private signal x_i (The length of each interval is shown on top.)

Proof. See Appendix D. □

A strategy profile in Theorem 7 is exactly the same as a strategy profile in Theorem 4 when setting $\rho = 1$. All the no-attack interval on the left of $\bar{x}^e$ has a measure of zero. The infinite switching disappears, and the equilibrium strategy is a monotone pure strategy. Figure 10 displays an equilibrium strategy profile for the case of weak substitutability.

As long as the introduced crowding-out is not severe, substitutability only affects the threshold $\bar{x}^e$ in the equilibrium outcome. In particular, the higher degree of substitutability leads to the lower threshold $\bar{x}^e$. We formally discuss the impact of substitutability in Section 5.

The constructed monotone pure-strategy equilibrium when the floating exchange rate is independent of the aggregate attacking size, i.e. $d(A) \equiv \bar{d}$ for some constant $\bar{d} \in \mathbb{R}_+$, is the same as the unique monotone pure-strategy equilibrium from the classical regime-change game.

Regarding the uniqueness, Theorem 5 still applies. So, the constructed monotone pure-strategy is again unique under a class of strategies with monotone aggregate action. Moreover, when the substitutability is weak, Theorem 8 states that the constructed monotone pure-strategy is globally unique.

Theorem 8 (Global Uniqueness when the Substitutability is Weak, i.e. $\rho = 1$)**.** *For $e > 0$, the strategy profile s^e described in equation (11) is a unique equilibrium strategy.*

Proof. See Appendix E. □

While the precision of the noise affects the switching frequency in the strong substitutability case ($\rho < 1$), the pattern of an equilibrium strategy is independent of the noise when the substitutability is weak ($\rho = 1$). The equilibrium strategy is a switching strategy, and the precision of the noise only affects the switching threshold. Again, the limiting precise private signal uniquely picks an equilibrium strategy from the common knowledge case. The convergent equilibrium strategy retains a monotone switching property. Speculators attack below a cutoff threshold and do not attack otherwise.

5 Model Implications

This paper extends standard regime-change games in a natural way. Inspired by the demand-driven price determination, we allow individual payoffs to vary with an aggregate attacking size. We analyze equilibrium outcomes under different information structures. Our main contribution is an analysis under private signals. We construct a pure-strategy Nash equilibrium and show that the specified equilibrium strategy is unique among all strategies with an aggregate action that is monotone in the fundamental.

When the substitutability force is strong, the pure-strategy Nash equilibrium infinitely switches in private signals. Speculators with better signals may attack, while those with worse signals may not. The switching between attacking and no attack becomes more rapid as noises get smaller. Despite this rapidly-switching behavior, the constructed equilibrium strategy has nice aggregate properties. A resulting aggregate attacking size is monotone in the realized but unobserved fundamental, and the fundamental switching threshold below which the regime switches does not depend on the precision of the noise.

In the limit of precise private signals, the equilibrium strategy converges to a monotone mixed-strategy equilibrium. That is, a mixed strategy is selected over a pure strategy over a range in the crisis region, where multiplicity is originally presented. This result may appear surprising at first because mixed-strategies are often thought to be non-robust under best-respond correspondences. However, we can reconcile this result using wisdom from Weinstein and Yildiz (2007). A private signal perturbation is a perturbation that rationalizes a mixed strategy over a given range of the fundamental.

The current setup nests standard regime-change games. The added substitutability affects the threshold below which the regime is overthrown. In particular, higher substitutability decreases incentives to attack and leads to a lower fundamental switching threshold as well as a lower probability of the peg breaking. If we can rank the substitutability, for example, by using pointwise dominance, we will also be able to order the switching threshold. We formalize the monotonicity property in the proposition below.

Proposition 9 (Monotonicity and Continuity)**.** *For any* $(d(\cdot), t)$ *and a given* $e \geq 0$*,*

1. *Continuity:* $\bar{x}^e$ *is continuous in* $(d(\cdot) + t, t)$ *endowed with the supremum norm. That is,* $\forall \epsilon > 0\ \exists \delta_1 > 0\ \exists \delta_2 > 0$ *such that* $\sup_{A \in [0,1]} \parallel (d_1(A) + t_1) - (d_2(A) + t_2) \parallel < \delta_1$ *and* $\mid t_1 - t_2 \mid < \delta_2 \Rightarrow \mid \bar{x}_1^e - \bar{x}_2^e \mid < \epsilon$

2. *Monotonicity:* $\bar{x}^e$ *is monotonically decreasing in* $(d(\cdot) + t, t)$ *under the pointwise dominance order. That is,* $d_1(A) + t_1 \geq d_2(A) + t_2$ *for all* $A \in [0, 1]$ *and* $t_1 \geq t_2 \Rightarrow \bar{x}_1^e \leq \bar{x}_2^e$*.*

Proof. See Appendix F $\square$

Corollary 10. *The fundamental switching threshold below which the exchange rate regime switches* $\hat{\theta} = \bar{x}^0$ *is monotonically decreasing in* $(d(\cdot) + t, t)$ *under the pointwise dominance order.*

Since the switching threshold is continuous and monotonic in the substitutability, we view our results as a natural extension of standard regime-change games.

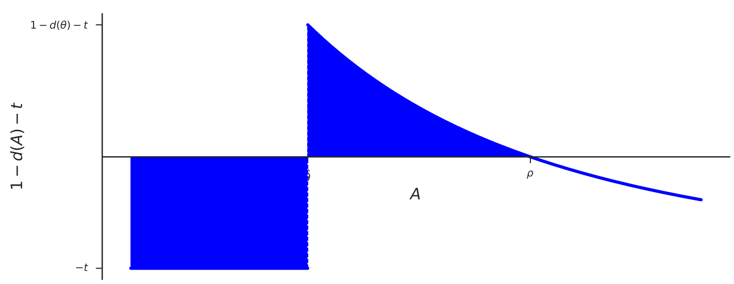


Figure 11: Individual attacking payoff as a function of aggregate attacking size

Figure 11 attempts to give an intuition on the fundamental switching threshold. We first note that the reward from a successful attack and the cost of attack are independent of the fundamental in our setup. These rewards and costs depend only on the aggregate attacking size. For such regime-switching games, the switching threshold is exactly the point where the expected reward from a successful attack covers the cost of attacking. In Figure 11, the threshold $\hat{\theta}$ locates exactly where the shaded area sums to zero.

5.1 Trade-Off between Returns and Liquidity

With a complete specification of an equilibrium strategy, this paper enables readers to calculate the cutoff $\bar{x}^e$ and the threshold $\hat{\theta}$ for any given level of noise, floating exchange function $d(\cdot)$, and the transaction cost t.

When $1 - d_1(A) - t_1$ crosses $1 - d_2(A) - t_2$ for some $A \in (0, 1)$, we move away from the pointwise dominance ordering of payoff functions. Suppose there are two currencies associated with $(d_1(A), t_1)$ and $(d_2(A), t_2)$, respectively. The first currency has potentially huge depreciation but also a bigger impact of trading size on the floating level, i.e. $1 - d_1(0) - t_1 > 1 - d_2(0) - t_2$ and $\frac{\partial d_1(A)}{\partial A} > \frac{\partial d_2(A)}{\partial A}$.

Figure 12 displays individual payoffs from attacking as a function of aggregate attacking size A. It is no longer obvious which of these currencies is more likely to come under pressure if they have similar fundamental levels. Nonetheless, our close-form formula of the threshold allows us to compare any currency pairs.

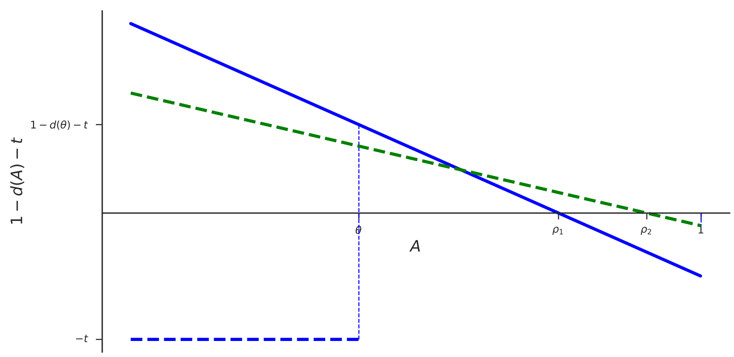
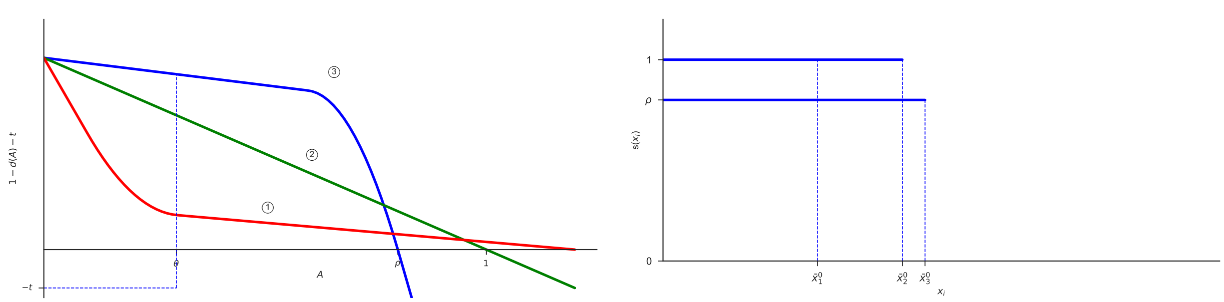


Figure 12: Individual attacking payoff as a function of aggregate attacking size. The blue solid line and the green dot line represent rewards from $(d_1(A), t_1)$ and $(d_2(A), t_2)$, respectively.

It now should be clear to the reader that the shape of the entire payoff function matters for the fundamental switching threshold. In Figure 13, the first and second payoff functions have weak substitutability $\rho = 1$. Individual rewards from attacking are non-negative even when everyone attacks the currency. However, both have lower switching thresholds $\hat{\theta} = \bar{x}^0$ than the third payoff functions with strong substitutability $\rho < 1$.



(a) Payoff Functions (b) Associated Switching Thresholds

Figure 13: Effects of payoff functions on switching thresholds

5.2 The Effect of External Selling Pressure

This section analyzes the effect of external selling pressure on the probability of regime switching. What we mean by external selling pressure is a mass $X \geq 0$ that will attack the currency in addition to these speculators. We assume that the size of this external selling X is common knowledge to every agent in the economy.

With only the complementarity force, external selling pressure strengthens the aggregate attacking coalition and increases the chance of regime switching. When the substitutability is present, it is no longer clear ex-ante how external selling pressure impacts the probability of abandoning the peg. On one hand, higher selling pressure helps coordination by increasing the size of aggregate attack. On the other hand, higher external selling pressure lowers individual net rewards from a successful attack. We show in this section that the coordination always dominates as long as the external selling pressure X is not too big.

Formally, suppose there exists an external selling pressure $X \in [0, \rho]$. The individual speculator's payoff is modified to

$$u(a_i, A, \theta) = a_i \left((1 - d(A + X)) \mathbf{1}_{A+X \geq \theta} - t \right). \tag{13}$$

Proposition 11 (External Selling Pressure)**.** *External selling pressure $X \in [0, \rho]$ always increases the fundamental switching threshold $\hat{\theta}$. That is, $\frac{\partial \hat{\theta}}{\partial X} > 0$.*

Proof. See Appendix G □

Without the crowding out when $d(\cdot) \equiv \bar{d}$ for some $\bar{d} \in \mathbb{R}_+$, external selling pressure X affects individual payoffs by effectively lowering the threshold of a successful attack from θ to $\theta - X$. Therefore, when individual payoffs do not depend on the aggregate attacking size, external selling pressure increases the fundamental switching threshold by exactly the size of external selling pressure.

When the aggregate attacking size affects the floating level: in particular, $\frac{\partial d(A)}{\partial A} > 0$, external selling pressure X helps coordination by increasing the size of aggregate attack and effectively lowering the switching threshold as before. However, external selling pressure X has another effect of lowering individual returns when an attack is successful.

Figure 14 shows how external selling pressure impacts individual payoffs for a given level of fundamental θ.

Proposition 11 shows that external selling pressure always moves the switching threshold to the right. The range of fundamental in which regime switches expands. The coordination force always dominates.

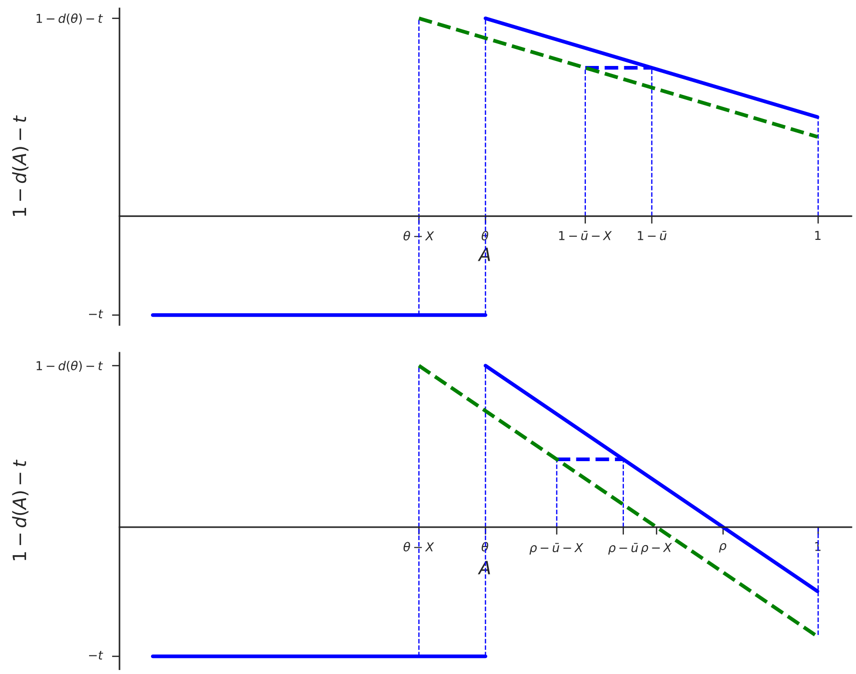


Figure 14: Individual payoffs as a function of aggregate attacking size A. The blue solid and green dash lines represent scenarios when there is no external selling pressure ($X = 0$) and when there is a positive external selling pressure ($X \in (0, \rho]$), respectively. The upper figure features the case of weak substitutability, and the lower figure features the strong substitutability case.

We first note the following: (1) for a given fundamental, the total aggregate selling pressure required to trigger the regime switching remains the same and (2) the fundamental threshold is exactly the threshold in which the expected individual attacking payoff is zero. Let $\hat{\theta}$ be the original switching threshold. At this $\hat{\theta}$, the positive payoffs from successful attack exactly offsets the cost of a failed attack. If external selling pressure crowds out speculators by no less than one to one, payoffs from attacking strictly go up because (1) an attack is successful more often and (2) the crowding out does not get more severe. At the equilibrium, external selling pressure must crowd out speculators by less than one to one. The switching threshold $\hat{\theta}$ must be higher.

5.3 Reevaluating Policy implications: The Effect of Quota

The substitutability in regime-change games highlights the need to reevaluate the effectiveness of various policies used in deterring speculative attacks. Previously, any policies that hinder the coordination among speculators will lower the probability of regime switching. It should now be clear to the reader that there is another dimension that policy makers should be aware of. In particular, how do these policies affect the substitutability among speculators' action?

For example, suppose there is a selling quota $Q \in [0,1]$ on how many speculators can sell in period 0. We can think of this quota as the level of speculation allowed in a given exchange rate market. This quota will alter the speculation in the following way. If the aggregate selling pressure A is greater than the imposed quota Q, there will be a rationing. An individual sell order will get filled with a probability $\frac{Q}{A}$. Otherwise, the quota is non-binding. That is, an attacking payoff is now modified as below:

$$u(a_i, A, \theta) = \begin{cases} a_i \left((1-d(A))\mathbf{1}_{A\geq\theta} - t\right) \text{ if } A \leq Q \\ a_i \frac{Q}{A}\left((1-d(Q))\mathbf{1}_{Q\geq\theta} - t\right) \text{ if } A > Q \end{cases}. \tag{14}$$

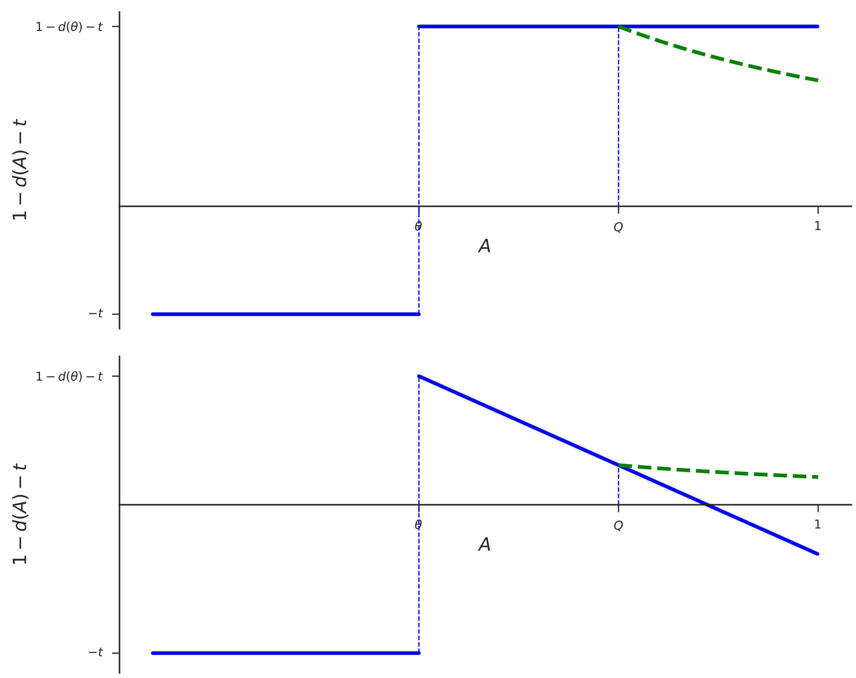


Figure 15: Individual payoffs as a function of aggregate attacking size A. The blue solid and green dash lines are when there is no quota ($Q = 0$) and when the quota is set at $Q \in [0,1)$, respectively. The upper figure represents the scenario in which the crowding out is not present, while the lower figure depicts the case with the strong substitutability.

Figure 15 illustrates the impact of a quota on individual attacking payoffs. Without substitutability, setting a quota dilutes positive individual payoffs and lowers individual incentives to attack. The threshold below which an attack is successful is lower. That is, the probability of regime switching decreases. When substitutability is present, an imposed quota has two counteracting effects. The quota weakens coordination but alleviates the crowding out. Depending on the severity of the crowding out and the level of an imposed quota, the fundamental switching threshold may now be higher, and the regime switches for a larger range of fundamentals.

Proposition 12 (The Effect of Quota)**.** *Consider any payoff functions $(d(\cdot), t)$. Let ρ be as defined in equation (2), and $\hat{\theta}$ and $\hat{\theta}_Q$ be fundamental thresholds below which regime switches when there is no quota and when the quota is set at $Q \in [0, 1]$, respectively. The following statements hold.*

1. *Without crowding out, i.e. $d(A) \equiv \bar{d}$ for some $\bar{d} \in \mathbb{R}_+$, an imposed quota $Q \in [0, 1]$ never expands the fundamental domain where the regime switches to floating. That is, $\hat{\theta}_Q \leq \hat{\theta}$.*

2. *When the crowding out is present, an imposed quota can expand the fundamental domain where the regime switches.*

    - *When the quota is quite restrictive, $Q \leq \hat{\theta}$, the fundamental switching threshold is lower, i.e. $\hat{\theta}_Q \leq \hat{\theta}$.*
    - *For an intermediate level of quota, $Q \in (\hat{\theta}, \rho)$, there exists $(d(\cdot), t)$ such that the fundamental switching threshold is higher, i.e. $\hat{\theta} < \hat{\theta}_Q$.*
    - *A quota that is bigger than the degree of substitutability, $Q \geq \rho$, does not affect the fundamental switching threshold. That is, $\hat{\theta}_Q = \hat{\theta}$.*

Proof. See Appendix H □

Proposition 12 states that imposing a quota always deters speculative attacks if substitutability is not present. With the crowding out, the effect of quota is more complicated. A quota alleviates the crowding out and ensures that the substitutability cannot be too severe.

Factoring in substitutability, an intermediate level of quota $Q \in (\hat{\theta}, \rho)$ can expand the domain where the regime switches to floating. Here, a quota alleviates the crowding out to a greater extent than hindering the coordination. A restrictive quota $Q \leq \hat{\theta}$ still deters an attack because the strict quota hinders coordination more than alleviates the crowding out. Similarly, a quota Q that is no less than the degree of substitutability ρ is not tight enough to lessen the substitutability from strong to weak. In this case, the fundamental switching threshold does not change.

The analysis in this section points out that policy makers must be careful when setting a level of an imposed quota. In general, we should evaluate any policies by looking at both their effects on coordination as well as their impacts on the crowding out.

6 Generalization and Other Applications

6.1 Generalization

So far, we present our setup in the context of speculative currency attacks. In fact, our analysis and results follow in a more generalized setup.

Consider any regime-change games with a measure one of players indexed by $i \in [0,1]$. Each player can choose to either attack ($a_i = 1$) or no attack ($a_i = 0$). $A = \int_0^1 a_i di$ represents an aggregate attacking size. Let θ be the strength of the fundamental. Again, the regime is overthrown whenever an aggregate attacking size exceeds the fundamental level, i.e. $A \geq \theta$.

Let $c(A) : [0,1] \to \mathbb{R}_+$ denotes the cost of attacking and $b(A) : [0,1] \to \mathbb{R}_+$ denotes the reward if the attack is successful. That is, the payoff to an individual player is

$$u(a_i, A, \theta) = a_i \left(b(A) \mathbf{1}_{A \geq \theta} - c(A) \right). \tag{15}$$

We make the following basic assumptions.

1. The reward and cost of attacking are continuous in an aggregate attacking size: $b(A)$ and $c(A)$ are continuous in A.

2. The potential reward from a successful attack is positive: $b(0) - c(0) > 0$.

The assumption of continuity simplifies the proof, while the second assumption of $b(0) - c(0) > 0$ ensures that the problem is interesting. It is worth trying to coordinate to overthrow the regime.

Assumption 13 (Single Crossing)**.** *A net payoff from attacking $b(A) - c(A)$ crosses 0 from above at most once for $A \in [0,1]$. Formally, one of the followings holds.*

1. *A net payoff from attacking is always positive: $b(A) - c(A) > 0$ for all $A \in [0,1]$.*

2. *There exists a unique $\rho \in [0,1]$ such that $b(\rho) - c(\rho) = 0$, $b(A) - c(A) > 0$ for all $A < \rho$ and $b(A) - c(A) < 0$ for all $A > \rho$.*

Assumption 14 (Monotonicity)**.** *The benefit if an attack is successful $b(A)$ is monotonically decreasing in an aggregate attacking size A, and the cost of attacking $c(A)$ is monotonically increasing in an aggregate attacking size A.*

Again, define the degree of substitutability ρ as follows:

$$\rho = \sup\{A \in [0,1] : b(A) - c(A) \geq c\}. \tag{16}$$

The equilibrium characterization for both the common knowledge case and the private information case along with the uniqueness theorem, the convergence proposition, the monotonicity and continuity proposition, the impact of external attacking pressure, and the effect of quota still hold if the monotonicity assumption in Assumption 14 is met.

Relaxing the monotonicity in Assumption 14 to the single crossing assumption in Assumption 13, the majority of results still hold including the equilibrium characterization for both the common knowledge case and the private information case, the uniqueness theorem, the convergence property, the monotonicity and continuity proposition, and the impact of external attacking pressure. A quota may never increase the chance of regime switching when the payoff from successful attack is not monotonically decreasing. Without the single crossing assumption (Assumption 13), we can no longer say anything about the uniqueness.

This section emphasizes that our results are indeed quite broad. There is nothing specific about the functional form of payoff functions. All conclusions remain true as long as the payoff function in the regime-change game is monotone. A formal discussion on the generalization of the results along with the associated proofs can be found in Appendix I.

6.2 Other Applications

This section considers how various applications can be set up as regime-change games with strategic substitutability and mentions potential implications for each application.

6.2.1 Venture Capital Investment

A startup has a liquidity need $I > 0$. There is a unit mass of venture capitalist and an exogenous mass $E \geq 0$ of existing stakeholders in this company. Each venture capitalist decides whether to buy a stake in this firm at a normalized price of 1. That is, each investor can take two actions: invest $(a_i = 1)$ or not invest $(a_i = 0)$. An aggregate funding raised, A, is equal to the total fraction of venture capitalists who decide to invest. That is, $A = \int_0^1 a_i di$.

If the startup raises enough capitals, i.e. $A \geq I$, the firm will realize the total return $R > 0$ that will be distributed equally among all stakeholders. Otherwise if the investment hurdle is not met, the firm will not distribute any returns. An individual venture capitalist's payoff is given as below:

$$u_i(a_i, A, \theta) = a_i \left(\frac{R}{A + E} \mathbf{1}_{A \geq I} - 1 \right). \tag{17}$$

We can analyze the trade-off between returns and the crowding out in this setting. We can compare which of the two startups would be more likely to raise enough fundings from venture capitalists: a startup with a higher total return R but with many existing stakeholders E that can dilute the individual payoff or the other startup with lower returns but less dilution.

Similar implications discussed earlier will follow. Here, having a quota on how many venture capitalists get to participate may increase the probability of a successful fundraising. In practice, we observe that venture capitalists are indeed concerned about this dilution effect, as they often impose restrictions on further participation in their proposed term sheets.

6.2.2 Debt Rollover

Consider a firm with a liquidity need θ in period 0. This firm has an expected positive cash flow in period 1. That is, it has a capacity to issue Z units of bonds that promise to pay a per-unit return of $R > 0$ in the next period. There is a continuum of investors indexed by $i \in [0,1]$. These investors can either invest 1 dollar in this bond ($a_i = 1$) or invest in risk-free alternatives ($a_i = 0$).

There is an external demand of $X \geq 0$ dollars for this bond. A period-0 unit price of the bond p is pinned down by the market clearing condition, i.e. $pZ = A + X$. If enough money is raised to meet the liquidity shortfall ($pZ \geq \theta$), the firm survives to period 1 and will pay the promised return. Otherwise, the firm will default.[6] The payoff for each individual investor is as follows:

$$\begin{aligned} u_i(a_i, A, \theta) &= a_i \left(\frac{R}{p} \mathbf{1}_{A \geq \theta} - 1 \right) \\ &= a_i \left(\frac{RZ}{A + X} \mathbf{1}_{A \geq \theta} - 1 \right). \end{aligned}$$

Given similar liquidity shortfalls, investors can decide which of the two firms are more likely to survive into the next period by looking at the total promised return RZ. The higher total promised payout RZ, the higher chance of a successful debt rollover. In equilibrium, it does not matter whether the firm increases the return per unit R or the debt pool Z because the equilibrium price p will adjust accordingly.

[6] The success of the debt rollover depends on investors' expectations. In Calvo (1988) and Cole and Kehoe (2000), investors expect some occurrences of defaults and require higher interest payments which indeed make a default optimal in some scenarios. Our setup follows He et al. (2015), and investors have beliefs about an aggregate demand of the bond. If investors expect a low demand of the bond, the firm will not raise enough money. Therefore, Investors will not buy the bond leading to a low aggregate bond demand and a default of the bond, respectively.

Any external demand $X \in (0, \rho]$ will always increase the probability of a successful debt rollover. Even though the crowding out lowers individual net returns, the coordination is a bigger problem.

Moreover, we can enrich the model by including the transaction cost $t(A) > 0$ associated with the sale of the bond and modifying the individual payoff to be:

$$u_i(a_i, A, \theta) = a_i \left(\frac{R}{p} \mathbf{1}_{A \geq \theta} - (1 + t(A)) \right) \tag{18}$$

$$= a_i \left(\frac{RZ}{A + X} \mathbf{1}_{A \geq \theta} - (1 + t(A)) \right). \tag{19}$$

There is a trade-off between potential returns and the liquidity. A firm with a high cash flow in the following period will have a capacity to promise high total payouts. However, if the trading of this corporate bond has a high transaction cost (potentially due to its illiquidity in the market), investors might find it more attractive to invest in another corporate bond with lower promised total payout but more readily tradable.

6.2.3 Overthrowing a Dictator

There is a unit mass of protestors indexed by $i \in [0, 1]$. These protestors can launch an attack to overthrow a dictator $(a_i = 1)$ or not $(a_i = 0)$. The strength of the dictatorship is given by the fundamental θ. Enough protestors must attack in order to successfully remove this dictator, that is $A = \int_i a_i di \geq \theta$.

The cost in launching an attack is $c > 0$. The payoff if the attack is successful is given by $b(A) > 0$ and may monotonically decrease with the size of an aggregate attack due to many reasons. One of the explanations is that these attackers may need to form a new coalition after removing the dictator. The more people in the coalition, the lower individual benefit from being a part of the group because of diverse interests. That is, an individual payoff is given by

$$u_i(a_i, A, \theta) = a_i \left(b(A) \mathbf{1}_{A \geq I} - c \right). \tag{20}$$

If a group of protestors has very diverse interests, high total benefits from overthrowing a regime may not be enough to encourage an attack. This setup implies that protests will be more likely among societies with similar interests. When an interest of each individual is too heterogenous, putting a limit on the number of seatings in the new parliament can potentially encourage an attack and might increase a chance of a successful removal of the dictator.

6.2.4 Further Discussion

Our setup can be applied generally to any coordination games. For example, take Morris and Shin (2004b)'s model of the run in asset markets. The incorporation of the substitutability feature can explain differential trading patterns between liquid and illiquid stocks. Here, we define illiquid securities as those with high price impacts from trading sizes. Results in our paper imply that individual strategies used in trading less liquid stocks are more complicated. Investors with worse news may now decide to hold on to illiquid assets while those with better news offload them. Intuitively, investors are aware that an execution of illiquid stocks incurs high slippage costs. In general, more extreme news will be needed to trigger the trade of more illiquid securities.

The impact of crowding out matters beyond applications to macroeconomics and financial economics. In the political economy setting, our model supports the concept of minimum winning coalition. Beyond the removal of a dictator, our model can explain formation patterns of treaty organizations or unions around the world. Additionally, the model speaks to urban economics. For example, extending the "big push" model, we can relate a city size with a probability of industrialization.

7 Conclusion

This paper studies how competition affects coordination games by analyzing regime-change games with individual payoffs that may vary with the size of an attacking coalition. It is essential to study such setup because, in real life applications, both benefits and costs of attacking often depend on the aggregate attacking size.

Contrast to previous literature on the interaction between strategic complementarity and strategic substitutability, the setup in this paper allows for the characterization of equilibrium outcomes when payoffs are subject to any degree of crowding out and presents the set of comparative statics with respect to the substitutability. Doing so, we offer a leading example of how to extend standard global games outside the supermodular setting.

Under common knowledge, there exists the crisis region with multiple equilibria. Under private information, higher-order beliefs complicate the equilibrium analysis, and standard global games resort to the iterated deletion of strictly dominated strategies to solve for equilibrium outcomes. However, with the strong substitutability, players' actions no longer admits a natural ordering, and the left dominance region vanishes. The paper proved the existence of equilibrium by constructing a strategy profile that can sustain a Nash equilibrium and showed that the constructed strategy is unique if we search over strategies that result in an aggregate attacking size that is monotonic in the underlying fundamental. Even in the generalized setup, taking away the common belief foundations appears to eliminate the indeterminacy of equilibrium outcomes. An insight from Morris and Shin (2000) remains.

Our setup highlights how the crowding out affects the coordination. Results presented in the paper make intuitive senses ex-post. Substitutability complicates an individual optimization problem and decreases agent's incentives to coordinate. The substitutability governs the pattern of individual equilibrium strategies as well as the fundamental switching threshold below which the attack is successful. In the limit of precise private noises, the specified equilibrium strategy converges to a monotone switching strategy. This convergent equilibrium strategy along with the associated fundamental threshold are continuous and monotone with respect to substitutability.

The enriched model presents the trade-off between potential rewards and the liquidity. Individuals should look at both potential total returns and how the crowding out alters individual payoffs. We revisited the effect of external coordinating pressure and found that an exogenous coordinating force helps the coordination more than crowds out returns. That is, the coordination force always dominates in the regime-switching setup. Regarding policy implications, the paper alerts policy makers to evaluate the impact of each policy by looking at both how it affects the coordination motive and how it alters the crowding out.

Accounting for the crowding out changes widely held intuitions on equilibrium outcomes. In the context of speculative currency attacks, selling a currency after receiving a sufficiently bad signal about the reserve may no longer be a dominant strategy; in the presence of substitutability, setting a quota on how many speculators can attack may increase the chance of abandoning the peg; currencies with potentially small depreciate but ample liquidity can be subject to more pressure than currencies with potentially large depreciation but low liquidity. Generally, the model in this paper is well suited to explain any interactions in group settings. Other applications include venture capital investment, debt rollover, overthrowing a dictator, etc.

The analysis in this paper open doors to many interesting questions. The most impending one is whether the equilibrium strategy is globally unique when the substitutability is strong. Secondly, the paper relies heavily on the assumption of uniform private noises. A robustness check of results with respect to other noise structures can be interesting. We conjecture that the existence, the uniqueness, and the limiting properties should continue to hold for more general noise distributions. Moreover, one can also study whether results continue to hold when the payoff function is generalized further. We are working to relax the assumption that benefits and costs depend only on the aggregate attacking size by allowing such benefits and costs to depend as well on the fundamental. Lastly, the paper focuses on the classical regime-change setting when players have a binary choice: to attack or not. An extension to multiple regimes can be interesting. Without substitutability, more regime choices should intuitively hinder coordination. With substitutability, a higher number of regimes may alleviate the crowding out, as players can derive higher payoffs by distributing their forces across different regimes. We leave a proper analysis for further research.

A Formal Equilibrium Characterization under Common Knowledge

A.1 Complete Information

For a given payoff function governed by $d(\cdot)$ and t, the set of equilibrium strategies $\Upsilon(\theta)$ is governed by

$$\Upsilon(\theta) = \begin{cases} \{\rho\} & ;\theta < 0 \\ \{0, \rho\} & ;0 \leq \theta \leq \rho \\ \{0\} & ;\rho < \theta \end{cases}, \tag{21}$$

for each ρ that is a solution to $1 - d(\rho) = t$.

A.2 Public Signal

The proposition below characterizes the set of equilibrium strategies when the noise is sufficiently small.

Proposition 15. *Consider a given payoff function governed by strictly increasing $d(\cdot)$ and t. Assume the noise of public signal is sufficiently small, i.e. $e < \frac{1}{2tM}$, when $M = \sup_{p \in [0,1]} \frac{|d'(p)|}{(1-d(p))^2}$, the set of equilibrium strategy $\Upsilon(y)$ is then given by*

$$\Upsilon(y) = \begin{cases} \{\rho\} & ;y < y_L \\ \{0, \delta(y), \rho\} & ;y_L \leq y \leq y_H(\rho) \\ \{0\} & ;y_H(\rho) < y \end{cases}, \text{ when} \tag{22}$$

$$y_L = 2e\left(\frac{1}{2} - \frac{t}{1-d(0)}\right), \quad y_H(\rho) = \begin{cases} 1 + 2e\left(\frac{1}{2} - \frac{t}{1-d(1)}\right) & ;\rho = 1 \\ \rho - e & ;\rho < 1 \end{cases}.$$

Remark 16. *$y_H(\rho)$ is continuous in ρ.*

Proof. <u>case I</u> $\rho = 1$:

No attack is an equilibrium if and only if $\mathbb{E}[u(1,0,\theta) \mid y] = (1-d(0))\mathbb{P}(\theta \leq 0 \mid y) - t \leq 0$, which holds whenever $y \geq 2e\left(\frac{1}{2} - \frac{t}{1-d(0)}\right) = y_L$. On the other hand, attack is an equilibrium if and only if $\mathbb{E}[u(1,1,\theta) \mid y] = (1-d(1))\mathbb{P}(\theta \leq 1 \mid y) - t \geq 0$, which holds whenever $y \leq 1 + 2e\left(\frac{1}{2} - \frac{t}{1-d(1)}\right)$.

A strictly mixed strategy $p \in (0,1)$ is sustainable in equilibrium if and only if $\mathbb{E}[u(1,p,\theta) \mid y] = (1-d(p))\mathbb{P}(\theta \leq p \mid y) - t = 0$.

Let $f(p) = \mathbb{P}(\theta \leq p \mid y)$, $g(p) = \frac{t}{1-d(p)}$, and $h(p) = f(p) - g(p)$. Note that

$$h(p) = \begin{cases} 1 - \frac{t}{1-d(p)} & ;y + e < p \\ \frac{p-(y-e)}{2e} - \frac{t}{1-d(p)} & ;y - e \leq p \leq y + e \\ -\frac{t}{1-d(p)} & ;p < y - e \end{cases}$$

For $p < y - e$, $h(p) < 0$. For $p > y + e$, $h(p) = 1 - \frac{t}{1-d(p)} > 1 - \frac{t}{1-d(1)} \geq 0$. The inequality follows from the strict monotonicity of $d(\cdot)$. Therefore, the solution to $h(p) = 0$ must be $p \in [y - e, y + e]$.

For $p \in [y-e, y+e]$, $\frac{\partial h(p)}{\partial p} = \frac{1}{2e} + \frac{td'(p)}{(1-d(p))^2} \geq \frac{1}{2e} - tM > 0$. That is, $h(p)$ is strictly increasing in p. $h(0) \leq 0$ if and only if $y \geq 2e\left(\frac{1}{2} - \frac{t}{1-d(0)}\right) = y_L$, while $h(1) \geq 0$ if and only if $y \leq 1 + 2e\left(\frac{1}{2} - \frac{t}{1-d(1)}\right) = y_H$. There is no intersection for $y \notin (y_L, y_H)$ and is exactly one intersection, namely $\delta(y)$, for $y \in (y_L, y_H)$ by the intermediate value theorem.

<u>case II</u> $\rho < 1$:

Recall that $\rho \in (0,1)$ is such that $1 - d(\rho) = t$. Similar to the previous case, no attack is an equilibrium if and only if $y \geq y_L$. Attack can never be sustained as an equilibrium as $(1-d(1))\mathbb{P}(\theta \leq p \mid y) - t \leq 1 - d(1) - t < 0$ for all $y \in \mathbb{R}$.

Any strictly-mixed strategy $p \in (0,1)$ is played in an equilibrium if and only if it is a solution to $h(p) = 0$. As before, we consider three different ranges of p with respect to y. For $p < y - e, h(p) < 0$. For $p > y + e, h(p) = 1 - \frac{t}{1-d(p)} = 0$ if and only if $p = \rho$. That is, attacking with probability ρ is an equilibrium strategy whenever $y < \rho - e$.

For $p \in [y-e, y+e], h(p) = \frac{p-(y-e)}{2e} - \frac{t}{1-d(p)}$, and thus, $\frac{\partial h(p)}{\partial p} > 0$. Note that for $p > \rho$, $h(p) = \frac{p-(y-e)}{2e} - \frac{t}{1-d(p)} \leq 1 - \frac{t}{1-d(p)} < 1 - \frac{t}{1-d(\rho)} < 0$. For $p = \rho$, $h(p) = 0$ if and only if

$y = \rho - e$. Therefore, an extra mixed strategy $\delta(y)$ must be in $(0, \rho)$.

Because $h(0) \leq 0$ if and only if $y \geq 2e\left(\frac{1}{2} - \frac{t}{1-d(0)}\right) = y_L$ and $h(\rho) \geq 0$ if and only if $y \leq \rho - e$, there is an extra intersection in addition to ρ, denoted by $\delta(y)$, if and only if $y \in (y_L, \rho - e]$.

□

B Proof of Theorem 4

Proof. Recall from (2) that ρ satisfies $1 - d(\rho) = t$. Let $\bar{u} \in (0, \rho)$ denotes a unique solution to

$$f(u) := \int_{\rho-u}^{\rho} (1 - d(A) - t)dA - (\rho - u)t = 0. \tag{23}$$

Note that such $\bar{u}$ exists and is unique since f is strictly increasing with $f(0) < 0$ and $f(\rho) > 0$. Next, we let

$$\bar{x}^e = -e + (1 + 2e)(\rho - \bar{u}). \tag{24}$$

Consider the following strategy

$$s(x_i) = \begin{cases} 0 & ; x_i \geq \bar{x}^e \\ 1 & ; \bar{x}^e - 2ek - 2e\rho \leq x_i < \bar{x}^e - 2ek, \quad k = 0, 1, 2, \ldots \\ 0 & ; \bar{x}^e - 2e(k+1) \leq x_i < \bar{x}^e - 2ek - 2e\rho, \quad k = 0, 1, 2, \ldots \end{cases}. \tag{25}$$

We now verify that $(s(x))_{x \in \mathbb{R}}$ is an equilibrium strategy profile. Assume that other players adopt the strategy $(s(x))_{x \in \mathbb{R}}$. Under a continuum of agents assumption, $A^s(\theta)$ is independent of a chosen action of player i and is given deterministically by

$$A(\theta) = \frac{\int_{\theta-e}^{\theta+e} \mathbf{1}_{\{s(u)=1\}} du}{2e} = \begin{cases} \rho & \theta < \bar{x}^e + e - 2e\rho \\ \frac{1}{2e}(\bar{x}^e + e - \theta) & \bar{x}^e + e - 2e\rho \leq \theta \leq \bar{x}^e + e \\ 0 & \theta > \bar{x}^e + e \end{cases}. \tag{26}$$

We observe that $A(\theta) - \theta$ is strictly decreasing over $\mathbb{R}$. Thus, there is a unique $\hat{\theta} \in \mathbb{R}$ such that $A(\hat{\theta}) = \hat{\theta}$. By equations (24) and (26), we have

$$A(\bar{x}^e + e - 2e\rho + 2e\bar{u}) - \bar{x}^e + e - 2e\rho + 2e\bar{u} = (\rho - \bar{u}) - (\rho - \bar{u}) = 0 \tag{27}$$

Therefore, a unique solution to $A(\theta) = \theta$ is

$$\hat{\theta} = \bar{x}^e + e - 2e\rho + 2e\bar{u}. \tag{28}$$

Note also that $\hat{\theta} \in (\bar{x}^e + e - 2e\rho, \bar{x}^e + e)$.

Next, we verify that s is indeed a best response correspondence. From equations (23), (24), and (28), it follows that

$$\begin{aligned}
F^A(\bar{x}^e) &= \frac{1}{2e}\int_{\bar{x}^e-e}^{\bar{x}^e+e}[(1-d(A(\theta)))\mathbf{1}_{A(\theta)\geq\theta} - t]d\theta \\
&= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\hat{\theta}}(1-d(A(\theta))-t)d\theta - \int_{\hat{\theta}}^{\bar{x}^e+e} t d\theta\right) \\
&= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e-2e\rho}(1-d(A(\theta))-t)d\theta + \int_{\bar{x}^e+e-2e\rho}^{\bar{x}^e+e-2e\rho+2e\bar{u}}(1-d(A(\theta))-t)d\theta\right) \\
&\quad - \frac{1}{2e}\left(\int_{\bar{x}^e+e-2e\rho+2e\bar{u}}^{\bar{x}^e+e} t d\theta\right) \\
&= \int_{\rho-\bar{u}}^{\rho}(1-d(A)-t)dA - (\rho-\bar{u})t = 0.
\end{aligned} \tag{29}$$

That is, a player observing signal $\bar{x}^e$ is indifferent between attacking and no attack. Since there is no reward beyond $\hat{\theta}$, a direct comparison yields $F(x_i) < F(\bar{x}^e) = 0$ for all $x_i > \bar{x}^e$. That is, no attack is dominant in such region, and $s(x_i) = 0$ for all $x_i > \bar{x}^e$.

For $x_i \in [\bar{x}^e - 2e\rho + 2e\bar{u}, \bar{x}^e)$, equation (28) implies that $x_i + e \geq \hat{\theta}$ and that

$$\begin{aligned}
F(x_i) &= \frac{1}{2e}\int_{x_i-e}^{x_i+e}(1-d(A(\theta)))\mathbf{1}_{A(\theta)\geq\theta}d\theta - t = \frac{1}{2e}\int_{x_i-e}^{\hat{\theta}}(1-d(A(\theta)))d\theta - t \\
&> \frac{1}{2e}\int_{\bar{x}^e-e}^{\hat{\theta}}(1-d(A(\theta)))d\theta - t = F(\bar{x}^e) = 0.
\end{aligned} \tag{30}$$

Attacking is dominant, and $s(x_i) = 1$ for $x_i \in [\bar{x}^e - 2e\rho + 2e\bar{u}, \bar{x}^e)$.

Next, for $x_i \in [\bar{x}^e - 2e\rho, \bar{x}^e - 2e\rho + 2e\bar{u})$, we have $x_i + e < \hat{\theta}$, so $A(\theta) > \theta$ for all $\theta \in [x_i - e, x_i + e]$. From equation (26), we also have that $A(\theta) \leq \rho$ for all $\theta \in [x_i - e, x_i + e]$. It follows that

$$F(x_i) = \frac{1}{2e}\int_{x_i-e}^{x_i+e}(1-d(A(\theta)))\mathbf{1}_{A(\theta)\geq\theta}d\theta - t \geq \frac{1}{2e}\int_{x_i-e}^{x_i+e}(1-d(\rho))d\theta - t = 0. \tag{31}$$

Attack is again dominant and $s(x_i) = 1$ for $x_i \in [\bar{x}^e - 2e\rho, \bar{x}^e - 2e\rho + 2e\bar{u})$.

Lastly, for $x_i \in (-\infty, \bar{x}^e - 2e\rho)$, $A(\theta) = \rho$ and $A(\theta) > \theta$ for all $\theta \in [x_i - e, x_i + e]$.

Therefore,

$$F(x_i) = \frac{1}{2e} \int_{x_i - e}^{x_i + e} (1 - d(\rho)) d\theta - t = t - t = 0. \tag{32}$$

That is, a player is indifferent in this signal interval.

From equations (29)-(32), we see that $(s(x))_{x \in \mathbb{R}}$ given by (25) is indeed a best response.

$\square$

C Proof of Theorem 5

Proof. First, note that there exists a unique $\hat{\theta}$ such that $A^s(\hat{\theta}) = \hat{\theta}$ due to the monotone assumption of A^s and $A^s(\theta) - \theta < 0$ (> 0) for sufficiently large (small) θ.

Second, we have that $A^s(\theta) \leq \rho$ for all θ. Otherwise, there exists θ_i with $A^s(\theta_i) > \rho$. By monotonicity of $A(\theta)$, $A^s(\theta) > \rho$ for all $\theta \geq \theta_i$. Therefore, for all $x_i \leq \theta_i - e$,

$$\begin{aligned} F(x_i) &= \frac{1}{2e} \left(\int_{x_i - e}^{x_i + e} (1 - d(A^s(\theta))) \mathbf{1}_{A^s(\theta) \geq \theta} d\theta \right) - t \\ &< \frac{1}{2e} \left(\int_{x_i - e}^{x_i + e} (1 - d(\rho)) d\theta \right) - t = 0. \end{aligned}$$

That is, $F(x_i) < 0$, and $s(x_i) = 0$ for all $x_i < \theta_i - e$, conflicting with Lemma 3.

Third, we show that there exists θ_0 with $A^s(\theta_0) = \rho$. If $A^s(\theta) < \rho$ for all θ, then, for all $x_i \leq \hat{\theta} - e$,

$$\begin{aligned} F(x_i) &= \frac{1}{2e} \left(\int_{x_i - e}^{x_i + e} (1 - d(A^s(\theta))) \mathbf{1}_{A^s(\theta) \geq \theta} d\theta \right) - t \\ &> \frac{1}{2e} \left(\int_{x_i - e}^{x_i + e} (1 - d(\rho)) d\theta \right) - t = 0. \end{aligned}$$

That is, $F(x_i) > 0$ for all $x_i \leq \hat{\theta} - e$. We have $\rho > A^s(\hat{\theta} - 2e) = 1 \geq \rho$, which is a contradiction.

Let $\theta_\rho = \sup\{\theta \in \mathbb{R}; A^s(\theta) = \rho\}$ and $\bar{x}^e = \sup\{x \in [\hat{\theta} - e, \hat{\theta} + e); \frac{\int_{x-e}^{\hat{\theta}} (1 - d(A^s(\theta))) d\theta}{2e} = t\}$.

θ_ρ is well-defined because (1) $A^s(\theta_0) = \rho$, (2) $A^s(\hat{\theta} + e) = 0$, and (3) $A^s(\theta)$ is continuous and weakly decreasing.

$\bar{x}^e$ is well-defined because $G(x) = \frac{\int_{\hat{\theta}-2e}^{\hat{\theta}} (1 - d(A^s(\theta))) d\theta}{2e}$ is continuous and weakly increasing with $G(\hat{\theta} - e) \geq t$ and $G(\hat{\theta} + e) = 0$. That is, $\bar{x}^e$ is the rightmost indifference point after which attacking becomes suboptimal.

We claim that $\theta_\rho < \hat{\theta}$. If $\theta_\rho \geq \hat{\theta}$, then for all $x_i > \hat{\theta} - e$,

$$
\begin{aligned}
F(x_i) &= \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e}(1-d(A^s(\theta)))\mathbf{1}_{A^s(\theta)\geq\theta}d\theta\right) - t \\
&= \frac{1}{2e}\left(\int_{x_i-e}^{\hat{\theta}}(1-d(A^s(\theta)))d\theta\right) - t \leq \frac{1}{2e}(\hat{\theta} - x_i + e)t - t < 0.
\end{aligned}
$$

That is, $F(x_i) < 0$ for all $x_i > \hat{\theta} - e$ and $0 = A^s(\hat{\theta}) \geq \rho$, which is a contradiction. It is then straightforward to check the following properties:

1. $F(x_i) < 0$ for all $x_i > \bar{x}^e$
2. $F(\bar{x}^e) = 0$
3. $F(x_i) > 0$ for all $x_i \in (\theta_\rho - e, \bar{x}^e)$
4. $F(x_i) = 0$ for all $x_i \leq \theta_\rho - e$
5. $A(\theta) = \rho$ for all $\theta \leq \theta_\rho$

For $F(x_i) > 0$ for all $x_i \in (\theta_\rho - e, \bar{x}^e)$, it is helpful to show (1) $F(x_i) > 0$ for all $x_i \in (\theta_\rho - e, \hat{\theta} - e)$ and (2) $F(x_i) > 0$ for all $x_i \in [\hat{\theta} - e, \bar{x}^e)$. The first equality follows from $A^s(\theta) < \rho$ for some $\theta \in (\theta_\rho, \hat{\theta}]$, while the second inequality follows directly as $F(\bar{x}^e) = 0$ and $F(x)$ is decreasing over $x \in [\hat{\theta} - e, \hat{\theta} + e)$.

For given $d(\cdot), t$, and e, $\hat{\theta}$ is unique, which leads to the unique $\bar{x}^e$. Consequently, $(A^s(\theta))_{\theta\in\mathbb{R}}$ and $(s(x_i))_{x_i\in\mathbb{R}}$ are uniquely determined. Now, we characterize the unique $\hat{\theta}$ for given $d(\cdot), t$, and e.

Recall that $A^s(\theta) = \frac{1}{2e}\int_{\theta-e}^{\theta+e} s(x_i)dx_i$ and is characterized by

$$
A^s(\theta) = \begin{cases} \rho & \theta \leq \theta_\rho \\ \frac{1}{2e}(\bar{x}^e + e - \theta) & \theta_\rho \leq \theta \leq \bar{x}^e + e \\ 0 & \theta \geq \bar{x}^e + e \end{cases}. \tag{33}
$$

Equation (33) and $A^s(\theta_\rho) = \rho$ imply that $\theta_\rho = \bar{x}^e + e - 2e\rho$. From $F(\bar{x}^e) = 0$, we have

$$\begin{aligned}
0 &= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e}(1-d(A^s(\theta)))\mathbf{1}_{A^s(\theta)\geq\theta} - t\right)d\theta \\
&= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\hat{\theta}}(1-d(A^s(\theta))-t) - \int_{\hat{\theta}}^{\bar{x}^e+e} t\right)d\theta \\
&= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e-2e\rho}(1-d(A^s(\theta))-t)d\theta + \int_{\bar{x}^e+e-2e\rho}^{\hat{\theta}}(1-d(A^s(\theta))-t)d\theta - \int_{\hat{\theta}}^{\bar{x}^e+e} t\right) \\
&= \int_{\frac{\bar{x}^e+e-\hat{\theta}}{2e}}^{\rho}(1-d(A)-t)dA - (\frac{\bar{x}^e+e-\hat{\theta}}{2e})t.
\end{aligned}$$

Let $\bar{u}$ be the unique value in $(0,\rho)$ such that $\int_{\rho-u}^{\rho}(1-d(A)-t)dA - (\rho-u)t = 0$. We have $\hat{\theta} = \bar{x}^e + e - 2e(\rho - \bar{u})$. From equation (33),

$$\hat{\theta} = A(\hat{\theta}) = A(\bar{x}^e + e - 2e\rho + 2e\bar{u}) = \rho - \bar{u}.$$

That is, $\hat{\theta}$ is uniquely determined. □

D Proof of Theorem 7

Proof. Denote $\bar{u} \in (0,1)$ a unique solution to

$$f(u) := \int_{1-u}^{1}(1-d(A)-t)dA - (1-u)t = 0.$$

Such $\bar{u}$ exists and is unique because $f(u)$ is strictly increasing $\forall u \in [0,1]$ with $f(0) < 0$ and $f(1) > 0$.

Consider the switching strategy

$$s(x_i) = \begin{cases} 1, & \text{if } x_i \leq \bar{x}^e \\ 0, & \text{if } x_i > \bar{x}^e \end{cases},$$

when $\bar{x}^e = -e + (1+2e)(1-\bar{u})$.

Assuming all other players use the described strategy $s(x_i)$, we have

$$A(\theta) = \int_{\theta-e}^{\theta+e} s(x_i)dx_i = \begin{cases} 1, & \text{if } \theta \leq \bar{x}^e - e \\ \frac{(\bar{x}^e+e)-\theta}{2e}, & \text{if } \theta \in [\bar{x}^e - e, \bar{x}^e + e] \\ 0, & \text{if } \theta \geq \bar{x}^e + e \end{cases}.$$

The expected payoff from attacking conditional on a private signal $\bar{x}^e$ is

$$\begin{aligned} F^A(\bar{x}^e) &= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e}(1-d(A^s(\theta)))\mathbf{1}_{A^s(\theta)\geq\theta}d\theta\right) - t \\ &= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\frac{\bar{x}^e+e}{1+2e}}(1-d(A^s(\theta))-t)d\theta - \int_{\frac{\bar{x}^e+e}{1+2e}}^{\bar{x}^e+e} td\theta\right) \\ &= \int_{1-\bar{u}}^{1}(1-d(A)-t)dA - (1-\bar{u})t = 0. \end{aligned}$$

The second equality follows from $A^s(\theta) \geq \theta \Leftrightarrow \frac{(\bar{x}^e+e)-\theta}{2e} \geq \theta \Leftrightarrow \frac{\bar{x}^e+e}{1+2e} \geq \theta$.

Since $A^s(\theta) - \theta$ is strictly decreasing, and $A^s(\theta) - \theta < (>)\ 0$ for sufficiently large (small) θ, there exists a unique θ satisfying $A^s(\theta) - \theta = 0$, denoted by $\hat{\theta}$. We then note that this $\hat{\theta} \in (\bar{x}^e - e, \bar{x}^e + e)$. Otherwise if $\hat{\theta} \leq \bar{x}^e - e$, $F^A(\bar{x}^e) = -t < 0$. Else if $\hat{\theta} \geq \bar{x}^e + e$, then $F^A(\bar{x}^e) = \int_0^1 (1-d(A)-t)dA > 0$. Both cases result in a contradiction.

Now consider $x_i < \bar{x}^e$. First, if $x_i \leq \hat{\theta} - e < \bar{x}^e$, then $F^A(x_i) \geq 1 - d(1) - t \geq 0$. If $\hat{\theta} - e < x_i \leq \bar{x}^e$, then

$$\begin{aligned} F^A(x_i) &= \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e}(1-d(A^s(\theta)))\mathbf{1}_{A^s(\theta)\geq\theta}d\theta\right) - t \\ &= \frac{1}{2e}\left(\int_{x_i-e}^{\hat{\theta}}(1-d(A^s(\theta)))d\theta\right) - t \\ &\geq \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\hat{\theta}}(1-d(A^s(\theta)))d\theta\right) - t \\ &= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e}(1-d(A^s(\theta)))\mathbf{1}_{A^s(\theta)\geq\theta}d\theta\right) - t \\ &= F^A(\bar{x}^e) = 0. \end{aligned}$$

That is, $F^A(x_i) \geq 0$, and hence $s(x_i) = 1$ is sustainable $\forall x_i \leq \bar{x}^e$. Similarly, we can prove that $F^A(x_i) < F^A(\bar{x}^e) = 0$ when $x_i > \bar{x}^e$, and hence $s(x_i) = 0$ is rationalizable $\forall x_i > \bar{x}^e$.

□

E Proof of Theorem 8

Proof. We use a similar technique from Goldstein and Pauzner (2005) to show that the equilibrium strategy s^e from equation (11) is a unique equilibrium strategy.

Suppose s^e be an equilibrium strategy when the noise is $e > 0$.

Let $\bar{x} = \sup\{x_i : F^s(x_i) \geq 0\}$. This $\bar{x}$ is well-defined because

1. $F^s(x_i) > 0$ for all $x_i < -e$, and
2. $F^s(x_i) < 0$ for all $x_i > 1 + e$.

Continuity of $F^s(x_i)$ implies that $F^s(\bar{x}) = 0$.

If s^e is a switching strategy with a threshold $\bar{x}$, we have $F^s(x_i) > 0$ for all $x_i < \bar{x}$ and $F^s(x_i) < 0$ for all $x_i > \bar{x}$. The aggregate attacking size is monotone in the fundamental. Since the characterized equilibrium from equation (11) is unique under a class of strategies with monotone aggregate action, s^e must coincide with that equilibrium strategy.

Otherwise, s^e is not a switching strategy. There exists $\underline{x}$ such that $\underline{x} = \sup\{x_i < \bar{x} : F^s(x_i) \leq 0\}$. We have that $F(x_i) < 0$ for all $x_i \in [\underline{x} - \epsilon, \underline{x})$ for some $\epsilon > 0$. That is, $A^s(\theta) < A^s(\underline{x} + e) \leq 1$ for all $\theta \in [\underline{x} + e - \epsilon, \underline{x} + e]$. Again, by the continuity of $F^s(\cdot)$, $F^s(\underline{x}) = 0$.

Let $d_{\bar{x}} = [\bar{x} - e, \bar{x} + e], d_{\underline{x}} = [\underline{x} - e, \underline{x} + e]$ and $I = d_{\bar{x}} \cap d_{\underline{x}}$.

case 1: $I = \emptyset$

We have that $A^s(\theta) = \frac{\bar{x}+e-\theta}{2e}$ for $[\bar{x} - e, \bar{x} + e]$, $A^s(\underline{x} + e) = 1$, and $A^s(\theta) < A^s(\underline{x} + e) = 1$ for $\theta \in [\underline{x} + e - \epsilon, \underline{x} + e)$.

If $A^s(\theta) > \theta$ for all $\theta \in d_{\underline{x}}$, then $F^s(\underline{x}) > 0$, a contradiction.

Otherwise, there exists $\hat{\theta}_{\underline{x}} \in d_{\underline{x}}$ such that $A^s(\hat{\theta}_{\underline{x}}) = \hat{\theta}_{\underline{x}}$. Let $\hat{\theta}_{\bar{x}} \in d_{\bar{x}}$ be such that $A^s(\hat{\theta}_{\bar{x}}) = \hat{\theta}_{\bar{x}}$. Because $A^s(\theta)$ is strictly decreasing in θ, we have that $\bar{x} + \underline{x} > \hat{\theta}_{\bar{x}} + \hat{\theta}_{\underline{x}}$ and

$A^s(\hat{\theta}_{\underline{x}}) < A^s(\hat{\theta}_{\bar{x}})$. Therefore,

$$
\begin{aligned}
0 &= F^s(\bar{x}) \\
&= \frac{1}{2e}\int_{\bar{x}-e}^{\bar{x}+e}\{(1-d(A^s(\theta)))\mathbf{1}_{A^s(\theta)>\theta} - t\}d\theta \\
&= \frac{1}{2e}\int_{\bar{x}-e}^{\hat{\theta}_{\bar{x}}}\{1-d(A^s(\theta)) - t\}d\theta + \frac{1}{2e}\int_{\hat{\theta}_{\bar{x}}}^{\bar{x}+e}(-t)d\theta \\
&< \frac{1}{2e}\int_{\hat{\theta}_{\underline{x}}}^{\underline{x}+e}\{1-d(A^s(\theta)) - t\}d\theta + \frac{1}{2e}\int_{\underline{x}-e}^{\hat{\theta}_{\underline{x}}}(-t)d\theta \\
&= \frac{1}{2e}\int_{\underline{x}-e}^{\underline{x}+e}(1-d(A^s(\theta)))\mathbf{1}_{A^s(\theta)>\theta} - td\theta \\
&= F^s(\underline{x}) \\
&= 0,
\end{aligned}
$$

leading to a contradiction as well.

case 2: $I \neq \emptyset$

Recall that $I = [\bar{x}^c, x^c]$. Let $d_{\bar{x}}\backslash I = [x^c, \bar{x}+e]$. Define $\bar{\theta} = \bar{x} + \underline{x} - \theta$. We will show that $\int_{x^c}^{\bar{x}+e}\{(1-d(A^s(\theta)))\mathbf{1}_{\{A^s(\theta)\geq\theta\}} - t\}d\theta < \int_{x^c}^{\bar{x}+e}\{(1-d(A^s(\bar{\theta})))\mathbf{1}_{\{A^s(\bar{\theta})\geq\bar{\theta}\}} - t\}d\theta$. Therefore, $0 = F^s(\bar{x}) < F^s(\underline{x}) = 0$, a contradiction.

We first note that the following is true.

1. $\mid \frac{\partial A^s(\bar{\theta})}{\partial\theta} \mid \leq \mid \frac{\partial A^s(\theta)}{\partial\theta} \mid = \frac{1}{2e}$ for all $\theta \in [x^c, \bar{x}]$

   This is because (1) $A^s(\theta) = \frac{\bar{x}+e-\theta}{2e}$ for all $\theta \in [x^c, \bar{x}]$ and (2) $A^s(\theta)$ changes at the fastest feasible rate of $\frac{1}{2e}$ when we are replacing attacking with no attack or vice a versa.

2. $\bar{\theta} < x^c$ for all $\theta \in [x^c, \bar{x}+e]$

   This is by the definition of $\bar{\theta}$.

3. For all $\theta \in I = [\bar{x}^c, x^c]$, $A^s(\theta) \geq A^s(x^c)$

   As we move to the left from x^c to $\bar{x}^c$, we replace agents with signals above $\bar{x}$ and do not attack with those with signal below $\underline{x}$ and may or may not attack. Therefore, the aggregate attacking size cannot decrease.

4. $\int_{x^c}^{\bar{x}+e}\{(1-d(A^s(\theta)))\mathbf{1}_{\{A^s(\theta)\geq\theta\}} - t\}d\theta \leq 0$

   Suppose $\int_{x^c}^{\bar{x}+e}\{(1-d(A^s(\theta)))\mathbf{1}_{\{A^s(\theta)\geq\theta\}} - t\}d\theta > 0$. Then, we know that $A^s(x^c) \geq x^c$ because otherwise $A^s(\theta) \leq \theta$ for all $\theta \in [x^c, \bar{x}+e]$, a contradiction.

Since $1 - d(A^s(\theta)) - t \geq 0$ for all θ and $A^s(\theta) \geq A^s(x^c) \geq x^c \geq \theta$ for all $\theta \in I = [\bar{x}^c, x^c]$, then we must have that $\int_I \{(1 - d(A^s(\theta)))\mathbf{1}_{\{A^s(\theta)\geq\theta\}} - t\}d\theta > 0$.

That is,

$$\begin{aligned}
0 &= F^s(\bar{x}) \\
&= \int_{x^c}^{\bar{x}+e} \{(1 - d(A^s(\theta)))\mathbf{1}_{\{A^s(\theta)\geq\theta\}} - t\}d\theta \\
&\quad + \int_I \{(1 - d(A^s(\theta)))\mathbf{1}_{\{A^s(\theta)\geq\theta\}} - t\}d\theta \\
&> 0 + 0 \\
&= 0.
\end{aligned}$$

Now, we are ready to prove that $\int_{x^c}^{\bar{x}+e} \{(1 - d(A^s(\theta)))\mathbf{1}_{\{A^s(\theta)\geq\theta\}} - t\}d\theta < \int_{x^c}^{\bar{x}+e} \{(1 - d(A^s(\bar{\theta})))\mathbf{1}_{\{A^s(\bar{\theta})\geq\bar{\theta}\}} - t\}d\theta$.

If $A^s(\bar{\theta}) \geq \bar{\theta}$ for all $\theta \in [x^c, \bar{x} + e]$, we know that $A^s(\theta) < A^s(\underline{x} + e) \leq 1$ for $\theta \in [\underline{x} + e - \epsilon, \underline{x} + e)$, i.e. $A^s(\bar{\theta}) < 1$ for $\theta \in [\bar{x} + e - \epsilon, \bar{x} + e)$. We will have that $\int_{x^c}^{\bar{x}+e} \{(1 - d(A^s(\bar{\theta})))\mathbf{1}_{\{A^s(\bar{\theta})\geq\bar{\theta}\}} - t\}d\theta > 0 \geq \int_{x^c}^{\bar{x}+e} \{(1 - d(A^s(\theta)))\mathbf{1}_{\{A^s(\theta)\geq\theta\}} - t\}d\theta$.

Otherwise, $A^s(\bar{\theta}) < \bar{\theta}$ for some $\theta \in [x^c, \bar{x} + e]$. Let $\hat{\theta}_{\underline{x}} \in d_{\underline{x}} = [\underline{x} - e, \bar{x}^c]$ such that $A^s(\hat{\theta}_{\underline{x}}) = \hat{\theta}_{\underline{x}}$. Let $\hat{\theta}_{\bar{x}} \in d_{\bar{x}}$ be such that $A^s(\hat{\theta}_{\bar{x}}) = \hat{\theta}_{\bar{x}}$. We again have that $\bar{x} + \underline{x} > \hat{\theta}_{\bar{x}} + \hat{\theta}_{\underline{x}}$.

If $\hat{\theta}_{\bar{x}} \in [\bar{x} - e, x^c)$, then

$$\begin{aligned}
\int_{x^c}^{\bar{x}+e} \{(1 - d(A^s(\theta)))\mathbf{1}_{\{A^s(\theta)\geq\theta\}} - t\}d\theta &= (\bar{x} + e - x^c)(-t) \\
&= (\bar{x}^c - (\underline{x} - e))(-t) \\
&< \int_{x^c}^{\bar{x}+e} \{(1 - d(A^s(\bar{\theta})))\mathbf{1}_{\{A^s(\bar{\theta})\geq\bar{\theta}\}} - t\}d\theta.
\end{aligned}$$

Otherwise, $\hat{\theta}_{\bar{x}} \in [x^c, \bar{x} + e]$. We know that $\bar{x} + \underline{x} > \hat{\theta}_{\bar{x}} + \hat{\theta}_{\underline{x}}$, $A^s(\hat{\theta}_{\bar{x}}) = \hat{\theta}_{\bar{x}} > \hat{\theta}_{\underline{x}} = A^s(\hat{\theta}_{\underline{x}})$, and $A^s(x^c) \leq A^s(\bar{x^c})$. Therefore,

$$\begin{aligned}
\int_{x^c}^{\bar{x}+e} \{(1 - d(A^s(\theta)))\mathbf{1}_{\{A^s(\theta)\geq\theta\}} - t\}d\theta &= \int_{x^c}^{\hat{\theta}_{\bar{x}}} \{1 - d(A^s(\theta)) - t\}d\theta + \int_{\hat{\theta}_{\bar{x}}}^{\bar{x}+e} (-t)d\theta \\
&< \int_{\hat{\theta}_{\underline{x}}}^{x^c} \{1 - d(A^s(\theta)) - t\}d\theta + \int_{\underline{x}-e}^{\hat{\theta}_{\underline{x}}} (-t)d\theta \\
&= \int_{x^c}^{\bar{x}+e} \{(1 - d(A^s(\bar{\theta})))\mathbf{1}_{\{A^s(\bar{\theta})\geq\bar{\theta}\}} - t\}d\theta.
\end{aligned}$$

$\square$

F Proof of Proposition 9

Proof. Consider $e \geq 0$ and a payoff function $d_1(\cdot)$ and t_1, we show the following:

1) Continuity: For any $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that for any payoff function $(d_2(\cdot), t_2)$ satisfying $\sup_{A\in[0,1]} \| (d_1(A)+t_1) - (d_2(A)+t_2) \| < \delta_1$ and $| t_1 - t_2 | < \delta_2$, we have $| \bar{x}_1^e - \bar{x}_2^e | < \epsilon$ when $\bar{x}_1^e$ and $\bar{x}_2^e$ are as defined in equation (9) corresponding to $(d_1(\cdot), t_1)$ and $(d_2(\cdot), t_2)$, respectively.

Fix $\epsilon > 0$, define the associated degree of substitutability ρ_1 and ρ_2 as in equation (2).

If $1 - d_1(1) - t_1 = \epsilon_1 > 0$ for some $\epsilon_1 > 0$, we have $\rho_1 = 1$ and $1 - d_2(1) - t_2 = 1 - d_1(1) - t_1 + (d_1(1) + t_1 - d_2(1) - t_2) \geq \epsilon_1 - \mid d_1(1) + t - 1 - d_2(1) - t_2 \mid > 0$ for $\delta_1 = \frac{\epsilon_1}{2}$. That is, ρ_2 must always be 1. In this case, $\rho_1 - \rho_2 = 0$.

Otherwise, there are $\rho_1, \rho_2 \in [0,1]$ such that $1 - d_1(\rho_1) = t_1$ and $1 - d_2(\rho_2) = t_2$. It follows that

$$
\begin{aligned}
\sup_{A\in[0,1]} \| d_1(A) + t_1 - (d_2(A) + t_2) \| < \delta_1 \Rightarrow & | d_1(\rho_1) + t_1 - d_2(\rho_1) - t_2 | < \delta_1, \\
& | d_1(\rho_2) + t_1 - d_2(\rho_2) - t_2 | < \delta_1 \\
\Rightarrow & 1 - \delta_1 < d_2(\rho_1) + t_2 < 1 + \delta_1, \\
& 1 - \delta_1 < d_1(\rho_2) + t_1 < 1 + \delta_1
\end{aligned}
$$

Let $m = \min_{A\in[0,1]} |d_1'(A)| > 0$, then by the Mean Value Theorem, there is ρ^* between ρ_1 and ρ_2

$$d_1(\rho_1) - d_1(\rho_2) = d_1'(\rho^*)(\rho_1 - \rho_2).$$

Thus,

$$| \rho_1 - \rho_2 | \leq \frac{\delta_1}{m}.$$

Therefore, for any $\epsilon > 0$, we can select δ_1 small enough that $| \rho_1 - \rho_2 | < \epsilon$.

Next, from the definition of $\bar{u}_1$ and $\bar{u}_2$, we have that

$$\int_{\rho_1-\bar{u}_1}^{\rho_1} (1 - d_1(A) - t_1) dA = \int_0^{\rho_1-\bar{u}_1} t_1 dA, \tag{34}$$

and

$$\int_{\rho_2-\bar{u}_2}^{\rho_2} (1 - d_2(A) - t_2) dA = \int_0^{\rho_2-\bar{u}_2} t_2 dA. \tag{35}$$

Without loss of generality, assume that $\rho_1 > \rho_2$. We can ensure that the selected δ is sufficiently small that $\rho_2 > \rho_1 - \bar{u}_1$. Subtracting equation (35) from equation (34), we have

$\int_{\rho_2}^{\rho_1}(1-d_1(A)-t_1)dA+\int_{\max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}^{\rho_2}(d_2(A)+t_2-d_1(A)-t_1)dA+(-1)^{i+1}\int_{\min(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}^{\max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}(1-d_i(A)-t_i)dA=((\rho_1-\bar{u}_1)-(\rho_2-\bar{u}_2))t_2+(\rho_1-\bar{u}_1)(t_1-t_2)$,where i is such that $\min(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)=\rho_i-\bar{u}_i$.

Let $M=1-d_i(\max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2))-t_i+t_2$, we have, by the triangle's inequality, the monotonicity of payoff functions, and the definition of supremum norm, that

$$
\begin{aligned}
&\mid (\rho_1-\bar{u}_1)-(\rho_2-\bar{u}_2)\mid M \\
&\quad\le\mid (-1)^{i+1}\int_{\min(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}^{\max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}(1-d_i(A)-t_i)dA-((\rho_1-\bar{u}_1)-(\rho_2-\bar{u}_2))t_2\mid \\
&\quad\le\mid \int_{\max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}^{\rho_2}(d_2(A)+t_2-d_1(A)-t_1)dA\mid \\
&\qquad+\mid \int_{\rho_2}^{\rho_1}(1-d_1(A)-t_1)dA\mid+\mid(\rho_1-\bar{u}_1)(t_1-t_2)\mid \\
&\quad\le \rho_2\delta_1+(\rho_1-\rho_2)(1-d_1(\rho_2)-t_1)+(\rho_1-\bar{u}_1)\delta_2
\end{aligned}
$$

That is,

$$
\begin{aligned}
\mid \bar{x}_1^e-\bar{x}_2^e\mid &=(2e+1)\mid(\rho_1-\bar{u}_1)-(\rho_2-\bar{u}_2)\mid \\
&\le(2e+1)\frac{\rho_2\delta_1+(\rho_1-\rho_2)(1-d_1(\rho_2)-t_1)+(\rho_1-\bar{u}_1)\delta_2)}{M} \\
&<\epsilon
\end{aligned}
$$

for sufficiently small δ_1 and δ_2.

2) Monotonicity: We want to show that $d_1(\cdot)+t_1\ge d_2(\cdot)+t_2$ and $t_1\ge t_2\Rightarrow \bar{x}_1^e\le\bar{x}_2^e$.

Assume $d_1(\cdot)+t_1\ge d_2(\cdot)+t_2$, we have that $\rho_1\le\rho_2$. It is left to show that $\rho_1-\bar{u}_1\le\rho_2-\bar{u}_2$. Suppose otherwise, i.e. $\rho_1-\bar{u}_1>\rho_2-\bar{u}_2$. This implies that $\rho_2\ge\rho_1\ge\rho_1-\bar{u}_1>\rho_2-\bar{u}_2$. Subtracting equation (35) from (34), we have

$$
\begin{aligned}
&\int_{\rho_1}^{\rho_2}(1-d_2(A)-t_2)dA+\int_{\rho_1-\bar{u}_1}^{\rho_1}(d_1(A)+t_1-d_2(A)-t_2)dA+ \\
&\qquad\int_{\rho_2-\bar{u}_2}^{\rho_1-\bar{u}_1}(1-d_2(A)-t_2)dA=(\rho_2-\bar{u}_2)(t_2-t_1)+\int_{\rho_2-\bar{u}_2}^{\rho_1-\bar{u}_1}(-t_1)dA \qquad (36)
\end{aligned}
$$

Now, we note that the RHS of equation (36) is negative. Now, consider the LHS of equation (36), $\int_{\rho_1}^{\rho_2}(1-d_2(A)-t_2)dA+\int_{\rho_1-\bar{u}_1}^{\rho_1}(d_1(A)+t_1-d_2(A)-t_2)dA+\int_{\rho_2-\bar{u}_2}^{\rho_1-\bar{u}_1}(1-d_2(A)-t_2)dA\ge\int_{\rho_1}^{\rho_2}(1-d_2(\rho_2)-t_2)dA+\int_{\rho_1-\bar{u}_1}^{\rho_1}(d_1(A)+t_1-d_2(A)-t_2)dA+\int_{\rho_2-\bar{u}_2}^{\rho_1-\bar{u}_1}(1-d_2(\rho_2)-t_2)dA\ge 0+0+0=0$. That is, the LHS of equation (36) is positive. We arrive at the contradiction. Therefore, $\rho_1-\bar{u}_1\le\rho_2-\bar{u}_2$ which implies $\bar{x}_1^e\le\bar{x}_2^e$ as desired.

□

G Proof of Proposition 11

Proof. Suppose there exists an external selling pressure $X \in [0, \rho]$. The individual payoff is now modified to

$$u_i(a_i, A, \theta) = a_i \left((1 - d(A + X)) \mathbf{1}_{A+X \geq \theta} - t \right).$$

It is straightforward to show that the following strategy described a Nash equilibrium of regime-change games with external selling pressure X.

$$s^e(x_i) = \begin{cases} 0 & ; x_i \geq \bar{x}^e \\ 1 & ; \bar{x}^e - 2ek - 2e(\rho - X) \leq x_i < \bar{x}^e - 2ek, \quad k = 0, 1, 2, \ldots \\ 0 & ; \bar{x}^e - 2e(k+1) \leq x_i < \bar{x}^e - 2ek - 2e(\rho - X), \quad k = 0, 1, 2, \ldots \end{cases}, \tag{37}$$

when $\bar{x}^e$ is given by

$$\bar{x}^e = -e + (1 + 2e)(\hat{\theta}_X - X) + X, \tag{38}$$

and $\hat{\theta}_X$ is the unique value $\theta \in (X, \rho)$ satisfying

$$\int_\theta^\rho (1 - d(A) - t) dA = \int_X^\theta t dA. \tag{39}$$

Let $G(\theta) = \int_\theta^\rho (1 - d(A) - t) dA - \int_X^\theta t dA$. Note that $G(X) > 0$, $G(\rho) < 0$, and $G'(\theta) = -(1 - d(\theta)) < 0$. Therefore, there exists a unique $\hat{\theta}_X$ such that $G(\hat{\theta}_X) = 0$ holds.

Note that $A^s(\theta) = \begin{cases} \rho - X & \text{for } \theta \leq \bar{x}^e - 2e(\rho - X) + e \\ \frac{\bar{x}^e + e - \theta}{2e} & \text{for } \theta \in [\bar{x}^e - 2e(\rho - X) + e, \bar{x}^e + e] \\ 0 & \text{for } \theta \geq \bar{x}^e + e \end{cases}$, and $A^s(\hat{\theta}_X) + X = \hat{\theta}_X$.

Note that $\hat{\theta}_X$ is the fundamental threshold below which the regime switches.

We are particularly interested in how X affects this fundamental threshold $\hat{\theta}_X$. Taking the derivative of equation (39) with respect to X using Leibniz's rule, we have

$$
\begin{aligned}
-(1-d(\hat{\theta}_X)-t)\frac{\partial\hat{\theta}_X}{\partial X} &= t[\frac{\partial\hat{\theta}_X}{\partial X}-1] \\
\frac{\partial\hat{\theta}_X}{\partial X} &= \frac{t}{1-d(\hat{\theta}_X)} > 0.
\end{aligned}
$$

That is, an external selling pressure always moves the fundamental switching threshold to the right. □

H Proof of Proposition 12

Proof. For a given payoff function $(d(\cdot),t)$. Let $\hat{\theta}$ and $\hat{\theta}_Q$ be fundamental thresholds when there is no imposed quota and when a quota is set at Q respectively.

case 1: $Q \leq \hat{\theta}$

Let s_Q be an equilibrium strategy when the quota is present. Observing $x_i \geq Q+e$, speculators know that the fundamental $\theta \geq Q$, and attack is never successful. $F^{s_Q}(x_i) < 0$. Therefore, $s_Q(x_i) = 0$ for all $x_i \geq Q+e$. Let $\bar{x}_Q^e = \sup\{x_i : F^{s_Q}(x_i) \geq 0\}$, we have that $\bar{x}_Q^e < Q+e$. In addition, we have that $A^{s_Q}(\theta) \leq Q$ for all $\theta \geq \bar{x}_Q + e - 2eQ$.

At the fundamental threshold $\hat{\theta}_Q$, we must have that $A^{s_Q}(\hat{\theta}_Q) = \hat{\theta}_Q$. We have the following:

$$
\begin{aligned}
\hat{\theta}_Q &= A^{s_Q}(\hat{\theta}_Q) \\
&\geq Q \\
&\geq A^{s_Q}(\theta) \text{ for all } \theta \geq \bar{x}_Q^e + e - 2eQ.
\end{aligned}
$$

We still restrict our search of strategy profiles over all those with monotone aggregate action. From $A^{s_Q}(\hat{\theta}_Q) \geq A^{s_Q}(\theta)$ for all $\theta \geq \bar{x}_Q^e + e - 2eQ$, we have that $\hat{\theta}_Q \leq \bar{x}_Q^e + e - 2eQ < Q + 2e(1-Q)$. In the limit of precise private signals, we have $\hat{\theta}_Q \leq Q \leq \hat{\theta}$. That is, $\hat{\theta}_Q \leq \hat{\theta}$ for all $Q \leq \hat{\theta}$.

When the quota is quite restrictive, the fundamental switching threshold is always lower than the threshold when the quota is not present.

case 2: $Q > \hat{\theta}$

$d(A) \equiv \bar{d}$ **for some** $\bar{d} \in \mathbb{R}_+$

First note that $1-\bar{d}-t > 0$ implies that $\frac{Q}{A}(1-\bar{d}-t) > 0$ for all $A \in [0,1]$. We know from Theorem 11 that $\hat{\theta}$ and $\hat{\theta}_Q$ are defined respectively as below.

$$\int_{\hat{\theta}}^{1}(1-\bar{d}-t)dA = \int_0^{\hat{\theta}} tdA$$
$$\int_Q^1 \frac{Q}{A}(1-\bar{d}-t)dA + \int_{\hat{\theta}_Q}^{Q}(1-\bar{d}-t)dA = \int_0^{\hat{\theta}_Q} tdA$$

Define $M(\theta) = \int_Q^1 \frac{Q}{A}(1-\bar{d}-t)dA + \int_{\theta}^{Q}(1-\bar{d}-t)dA - \int_0^{\theta} tdA$. We have that $\frac{\partial M(\theta)}{\partial \theta} < 0$ and $F(\hat{\theta}) < 0$, therefore we must have that $\hat{\theta}_Q < \hat{\theta}$.

Ambiguity Effects of Quota

There exists payoff functions $(d(\cdot), t)$ such that $\frac{Q}{A}(1-d(Q)-t) \geq 1-d(A)-t$ for all $A \in [Q,1]$, and the inequality is strict over a non-zero measure interval that is a subset of $[Q,1]$.

If $1-d(A)-t \geq 0$ for all $A \in [0,1]$, $\rho = 1$ and $\hat{\theta}$ is such that

$$\int_{\hat{\theta}}^{1}(1-d(A)-t)dA = \int_0^{\hat{\theta}} tdA.$$

Define $M(\theta) = \int_Q^1 \frac{Q}{A}(1-d(A)-t)dA + \int_{\theta}^{Q}(1-d(A)-t)dA - \int_0^{\theta} tdA$. Again, we have $\frac{\partial M(\theta)}{\partial \theta} < 0$. However, here $M(\hat{\theta}) > \int_{\hat{\theta}}^1(1-d(A)-t)dA - \int_0^{\hat{\theta}} tdA = 0$. Therefore, $\hat{\theta}_Q$ such that $M(\hat{\theta}_Q) = 0$ must be greater than $\hat{\theta}$. That is, $\hat{\theta} < \hat{\theta}_Q$.

If $1-d(1)-t < 0$, then $\rho < 1$. Now $\hat{\theta}$ is defined from Theorem 4 as below:

$$\int_{\hat{\theta}}^{\rho}(1-d(A)-t)dA = \int_0^{\hat{\theta}} tdA. \tag{40}$$

There are 2 cases.

1. $Q < \rho$ or $1-d(Q)-t > 0$

   Define $M(\theta) = \int_Q^1 \frac{Q}{A}(1-d(A)-t)dA + \int_{\theta}^{Q}(1-d(A)-t)dA - \int_0^{\theta} tdA$. Again, we have $\frac{\partial M(\theta)}{\partial \theta} < 0$. However, here $M(\hat{\theta}) > \int_{\hat{\theta}}^{\rho} \frac{Q}{A}(1-d(A)-t)dA - \int_0^{\hat{\theta}} tdA \geq \int_{\hat{\theta}}^{\rho}(1-d(A)-t)dA - \int_0^{\hat{\theta}} tdA = 0$. Therefore, $\hat{\theta}_Q$ such that $M(\hat{\theta}_Q) = 0$ must be greater than $\hat{\theta}$. That is, $\hat{\theta} < \hat{\theta}_Q$.

2. $Q \geq \rho$ or $1-d(Q)-t \leq 0$

   The fundamental threshold $\hat{\theta}_Q$ is then defined by the following equation:

$$\int_{\hat{\theta}Q}^{1}(1-d(A)-t)dA = \int_0^{\hat{\theta}} tdA,$$

which is exactly the same as equation (40). That is, $\hat{\theta}_Q = \hat{\theta}$.

$\square$

I Formal Results for the Generalized Case

I.1 Complete Information

For a given payoff function governed by $b(\cdot)$ and $c(\cdot)$, the set of equilibrium strategies $\Upsilon(\theta)$ is governed by

$$\Upsilon(\theta) = \begin{cases} \{\rho\} & ;\theta < 0 \\ \{0, \rho\} & ;0 \leq \theta \leq \rho \\ \{0\} & ;\rho < \theta \end{cases}, \tag{41}$$

for each ρ that is a solution to $b(\rho) - c(\rho) = 0$.

I.2 Public Signal

The proposition below characterizes the set of equilibrium strategies when the noise is sufficiently small.

Proposition 17. *Consider a given payoff function governed by $b(A)$ and $c(A)$ when $b(A) - c(A)$ satisfy the single crossing assumption in Assumption 13. Assume the noise of public signal is sufficiently small, i.e. $e < \frac{1}{2}M$, when $M = \sup_{p \in [0,1]} \frac{b(p)c'(p) - c(p)b'(p)}{(b(p))^2}$, the set of equilibrium strategy $\Upsilon(y)$ is then given by*

$$\Upsilon(y) = \begin{cases} \{\rho\} & ;y < y_L \\ \{0, \delta(y), \rho\} & ;y_L \leq y \leq y_H(\rho) \\ \{0\} & ;y_H(\rho) < y \end{cases}, \text{ when} \tag{42}$$

$$y_L = 2e\left(\frac{1}{2} - \frac{c(o)}{b(0)}\right), \quad y_H(\rho) = \begin{cases} 1 + 2e\left(\frac{1}{2} - \frac{c(1)}{b(1)}\right) & ;\rho = 1 \\ \rho - e & ;\rho < 1 \end{cases}.$$

Remark 18. *$y_H(\rho)$ is continuous in ρ.*

Proof. Recall that either the payoff from successful attack is never negative ($\rho = 1$) or ρ is exactly the aggregate attacking size such that the payoff from successful attack, $b(\rho) - c(\rho)$, is zero.

case I $\rho = 1$:

No attack is an equilibrium strategy if and only if

$$\mathbb{E}[u(1,0,\theta) \mid y] = b(0)\mathbb{P}(\theta \leq 0 \mid y) - c(0) \leq 0,$$

which holds whenever $y \geq 2e\left(\frac{1}{2} - \frac{c(0)}{b(0)}\right) = y_L$.

On the other hand, attacking is an equilibrium strategy if and only if

$$\mathbb{E}[u(1,1,\theta) \mid y] = b(1)\mathbb{P}(\theta \leq 1 \mid y) - c(1) \geq 0,$$

which holds whenever $y \leq 1 + 2e\left(\frac{1}{2} - \frac{c(1)}{b(1)}\right) = y_H$.

A strictly mixed strategy $p \in (0,1)$ is an equilibrium strategy if and only if

$$\mathbb{E}[u(1,p,\theta) \mid y] = b(p)\mathbb{P}(\theta \leq p \mid y) - c(p) = 0.$$

Let $f(p) = \mathbb{P}(\theta \leq p \mid y)$, $g(p) = \frac{c(p)}{b(p)}$, and $h(p) = f(p) - g(p)$. Note that

$$h(p) = \begin{cases} 1 - \frac{c(p)}{b(p)} & ; y + e < p \\ \frac{p-(y-e)}{2e} - \frac{c(p)}{b(p)} & ; y - e \leq p \leq y + e \\ -\frac{c(p)}{b(p)} & ; p < y - e \end{cases}.$$

For $p < y - e$, $h(p) < 0$. For $p > y + e$, $h(p) = 1 - \frac{c(p)}{b(p)} \geq 1 - \frac{c(1)}{b(1)} \geq 0$. The inequality follows from the single crossing property of the payoff $b(\cdot) - c(\cdot)$. For $p < 1$, the solution to $h(p) = 0$ must be $p \in [y - e, y + e]$.

For $p \in [y - e, y + e]$, $\frac{\partial h(p)}{\partial p} = \frac{1}{2e} - \frac{\partial(\frac{c(p)}{b(p)})}{\partial p} \geq \frac{1}{2e} - M > 0$. That is, $h(p)$ is strictly increasing in p. $h(0) \leq 0$ if and only if $y \geq 2e\left(\frac{1}{2} - \frac{c(0)}{b(0)}\right) = y_L$, while $h(1) \geq 0$ if and only if $y \leq 1 + 2e\left(\frac{1}{2} - \frac{c(1)}{b(1)}\right) = y_H$. There is no intersection for $y \notin (y_L, y_H)$ and is exactly one intersection, namely $\delta(y)$, for $y \in (y_L, y_H)$ by the intermediate value theorem.

case II $\rho < 1$:

Recall that $\rho \in (0,1)$ is such that $b(\rho) - c(\rho) = 0$. Similar to the previous case, no attack is an equilibrium strategy if and only if $y \geq y_L$. Attack can never be sustained an equilibrium strategy because $b(1)\mathbb{P}(\theta \leq p \mid y) - c(1) \leq b(1) - c(1) < 0$ for all $y \in \mathbb{R}$.

Any strictly-mixed strategy $p \in (0,1)$ is played in an equilibrium if and only if it is a solution to $h(p) = 0$. As before, we consider three different ranges of p with respect to y. For $p < y - e, h(p) < 0$. For $p > y + e, h(p) = 1 - \frac{c(p)}{b(p)} = 0$ if and only if $p = \rho$. That is, attacking with probability ρ is an equilibrium strategy whenever $y < \rho - e$.

For $p \in [y-e, y+e], h(p) = \frac{p-(y-e)}{2e} - \frac{c(p)}{b(p)}$, and thus, $\frac{\partial h(p)}{\partial p} > 0$. Note that for $p > \rho$, $h(p) = \frac{p-(y-e)}{2e} - \frac{c(p)}{b(p)} \leq 1 - \frac{c(p)}{b(p)} < 1 - \frac{c(\rho)}{b(\rho)} < 0$. For $p = \rho$, $h(p) = 0$ if and only if $y = \rho - e$. Therefore, an extra mixed strategy $\delta(y)$ must be in $(0, \rho)$.

Because $h(0) \leq 0$ if and only if $y \geq 2e\left(\frac{1}{2} - \frac{c(0)}{b(0)}\right) = y_L$ and $h(\rho) \geq 0$ if and only if $y \leq \rho - e$, there is an extra intersection in addition to ρ, denoted by $\delta(y)$, if and only if $y \in (y_L, \rho - e]$.

Note: The proof just uses the definition of ρ along with the single crossing assumption (Assumption 13). □

I.3 Private Information

Lemma 19 (Infinite-Switching Pure-Strategy)**.** *For $b(\cdot)$ and $c(\cdot)$ such that $\rho < 1$, consider any pure-strategy Nash equilibrium profile $s : \mathbb{R} \to \{0, 1\}$. If s is constant over any interval $I = (\underline{x}, \bar{x})$ with $\bar{x} \leq 0$, then $|I| = \bar{x} - \underline{x} < 4e$.*

Proof. Suppose there exists a pure-strategy Nash equilibrium profile s and an interval $I = (\underline{x}, \bar{x})$ with $\bar{x} \leq 0$ and $|I| \geq 4e$. Let $x_m = \frac{\underline{x}+\bar{x}}{2}$. If $s(x) \equiv 0$ on I, then $A(\theta) = 0 \geq \theta$ for all $\theta \in [x_m - e, x_m + e]$ and $F^s(x_m) = b(0) - c(0) > 0$. Attacking yields strictly higher payoffs than no attack, contradicting the fact that s is an equilibrium strategy. Similarly, if $s(x) \equiv 1$ on I, then $A(\theta) = 1 \geq \theta$ for all $\theta \in [x_m - e, x_m + e]$. Therefore, $F^s(x_m) = b(1) - c(1) < b(\rho) - c(\rho) = 0$. Deviating to no attack yields strictly higher payoff, again contradicting the fact that s is an equilibrium strategy. □

Theorem 20 (Existence of Pure-Strategy Equilibrium)**.** *The following pure-strategy profile describes a Nash equilibrium of regime-change games .*

$$s^e(x_i) = \begin{cases} 0 & ; x_i \geq \bar{x}^e \\ 1 & ; \bar{x}^e - 2ek - 2e\rho \leq x_i < \bar{x}^e - 2ek, \quad k = 0, 1, 2, \dots \\ 0 & ; \bar{x}^e - 2e(k+1) \leq x_i < \bar{x}^e - 2ek - 2e\rho, \quad k = 0, 1, 2, \dots \end{cases}, \tag{43}$$

when $\bar{x}^e$ is given by

$$\bar{x}^e = -e + (1 + 2e)(\rho - \bar{u}), \tag{44}$$

and $\bar{u}$ is the unique value $u \in (0, \rho)$ satisfying

$$\int_{\rho-u}^{\rho} (b(A) - c(A))dA = \int_0^{\rho-u} c(A)dA.$$

Proof. Recall from (16) that either $\rho = 1$ and $b(A) - c(A) \geq 0$ for all $A \in [0,1]$ or $\rho < 1$ such that $b(\rho) - c(\rho) = 0$ and $b(A) - c(A) \geq 0$ for all $A \in [0, \rho)$

case 1: $\rho < 1$ Let $\bar{u} \in (0, \rho)$ denotes a unique solution to

$$f(u) := \int_{\rho-u}^{\rho} (b(A) - c(A))dA + \int_{0}^{\rho-u} (0 - c(A))dA = 0. \tag{45}$$

Note that such $\bar{u}$ exists and is unique since f is strictly increasing u with $f(0) < 0$ and $f(\rho) > 0$. Next, we let

$$\bar{x}^e = -e + (1 + 2e)(\rho - \bar{u}). \tag{46}$$

Consider the following strategy

$$s(x_i) = \begin{cases} 0 & ; x_i \geq \bar{x}^e \\ 1 & ; \bar{x}^e - 2ek - 2e\rho \leq x_i < \bar{x}^e - 2ek, \quad k = 0, 1, 2, \ldots \\ 0 & ; \bar{x}^e - 2e(k+1) \leq x_i < \bar{x}^e - 2ek - 2e\rho, \quad k = 0, 1, 2, \ldots \end{cases}. \tag{47}$$

We now verify that $(s(x))_{x \in \mathbb{R}}$ is an equilibrium strategy profile. Assume that other players adopt the strategy $(s(x))_{x \in \mathbb{R}}$. Under a continuum of agents assumption, $A^s(\theta)$ is independent of a chosen action of player i and is given deterministically by

$$A(\theta) = \frac{\int_{\theta-e}^{\theta+e} \mathbf{1}_{\{s(u)=1\}} du}{2e} = \begin{cases} \rho & \theta < \bar{x}^e + e - 2e\rho \\ \frac{1}{2e}(\bar{x}^e + e - \theta) & \bar{x}^e + e - 2e\rho \leq \theta \leq \bar{x}^e + e \\ 0 & \theta > \bar{x}^e + e \end{cases}. \tag{48}$$

We observe that $A(\theta) - \theta$ is strictly decreasing over $\mathbb{R}$. Thus, there is a unique $\hat{\theta} \in \mathbb{R}$ such that $A(\hat{\theta}) = \hat{\theta}$. By (46) and (48), we have

$$A(\bar{x}^e + e - 2e\rho + 2e\bar{u}) - \bar{x}^e + e - 2e\rho + 2e\bar{u} = (\rho - \bar{u}) - (\rho - \bar{u}) = 0 \tag{49}$$

Therefore, a unique solution to $A(\theta) = \theta$ is

$$\hat{\theta} = \bar{x}^e + e - 2e\rho + 2e\bar{u}. \tag{50}$$

Note also that $\hat{\theta} \in (\bar{x}^e + e - 2e\rho, \bar{x}^e + e)$.

Next, we verify that s is indeed a best response correspondence. From equations (45), (46), and (50), it follows that

$$
\begin{aligned}
F^A(\bar{x}^e) &= \frac{1}{2e}\int_{\bar{x}^e-e}^{\bar{x}^e+e}(b(A(\theta))\mathbf{1}_{A(\theta)\geq\theta} - c(A(\theta))d\theta \\
&= \frac{1}{2e}\int_{\bar{x}^e-e}^{\hat{\theta}}(b(A(\theta)) - c(A(\theta)))d\theta + \frac{1}{2e}\int_{\hat{\theta}}^{\bar{x}^e+e}(0 - c(A(\theta)))d\theta \\
&= \frac{1}{2e}\int_{\bar{x}^e-e}^{\bar{x}^e+e-2e\rho}(b(A(\theta)) - c(A(\theta)))d\theta + \frac{1}{2e}\int_{\bar{x}^e+e-2e\rho}^{\bar{x}^e+e-2e\rho+2e\bar{u}}(b(A(\theta)) - c(A(\theta)))d\theta \\
&\quad + \frac{1}{2e}\int_{\bar{x}^e+e-2e\rho+2e\bar{u}}^{\bar{x}^e+e}(0 - c(A(\theta)))d\theta \\
&= (1-\rho)(b(\rho) - c(\rho)) + \int_{\rho-u}^{\rho}(b(A) - c(A))dA + \int_{0}^{\rho-u}(0 - c(A))dA \\
&= 0.
\end{aligned}
\tag{51}
$$

That is, a player observing signal $\bar{x}^e$ is indifferent between attacking and no attack. Since there is no reward beyond $\hat{\theta}$, a direct comparison yields $F(x_i) < F(\bar{x}^e) = 0$ for all $x_i > \bar{x}^e$. That is, no attack is dominant in such region, and $s(x_i) = 0$ for all $x_i > \bar{x}^e$.

For $x_i \in [\bar{x}^e - 2e\rho + 2e\bar{u}, \bar{x}^e)$, equation (50) implies that $x_i + e \geq \hat{\theta}$ and that

$$
\begin{aligned}
F(x_i) &= \frac{1}{2e}\int_{x_i-e}^{x_i+e}(b(A(\theta))\mathbf{1}_{A(\theta)\geq\theta} - c(A(\theta))d\theta \\
&= \frac{1}{2e}\int_{x_i-e}^{\hat{\theta}}(b(A(\theta)) - c(A(\theta)))d\theta + \frac{1}{2e}\int_{\hat{\theta}}^{x_i+e}(0 - c(A(\theta)))d\theta \\
&> \frac{1}{2e}\int_{\bar{x}^e-e}^{\hat{\theta}}(b(A(\theta)) - c(A(\theta)))d\theta + \frac{1}{2e}\int_{\hat{\theta}}^{\bar{x}^e+e}(0 - c(A(\theta)))d\theta \\
&= F(\bar{x}^e) = 0.
\end{aligned}
\tag{52}
$$

Attacking is dominant, and $s(x_i) = 1$ for $x_i \in [\bar{x}^e - 2e\rho + 2e\bar{u}, \bar{x}^e)$.

Next, for $x_i \in [\bar{x}^e - 2e\rho, \bar{x}^e - 2e\rho + 2e\bar{u})$, we have $x_i + e < \hat{\theta}$, so $A(\theta) > \theta$ for all $\theta \in [x_i - e, x_i + e]$. From equation (48), we also have that $A(\theta) \leq \rho$ for all $\theta \in [x_i - e, x_i + e]$. It follows that

$$
\begin{aligned}
F(x_i) &= \frac{1}{2e}\int_{x_i-e}^{x_i+e}(b(A(\theta))\mathbf{1}_{A(\theta)\geq\theta} - c(A(\theta)))d\theta \\
&\geq \frac{1}{2e}\int_{x_i-e}^{x_i+e}(b(\rho) - c(\rho))d\theta = 0.
\end{aligned}
\tag{53}
$$

Attack is again dominant and $s(x_i) = 1$ for $x_i \in [\bar{x}^e - 2e\rho, \bar{x}^e - 2e\rho + 2e\bar{u})$.

Lastly, for $x_i \in (-\infty, \bar{x}^e - 2e\rho)$, $A(\theta) = \rho$ and $A(\theta) > \theta$ for all $\theta \in [x_i - e, x_i + e]$.

Therefore,

$$F(x_i) = \frac{1}{2e}\int_{x_i-e}^{x_i+e}(b(\rho)-c(\rho))du = 0. \tag{54}$$

That is, a player is indifferent in this signal interval.

From equations (51)-(54), we see that $(s(x))_{x\in\mathbb{R}}$ given by (47) is indeed the best response.

case 2: $\rho = 1$

Denote $\bar{u} \in (0,1)$ a unique solution to

$$f(u) := \int_{1-u}^{1}(b(A)-c(A))dA - \int_{0}^{1-u}c(A)dA = 0.$$

Such $\bar{u}$ exists and is unique because $f(u)$ is strictly increasing $\forall u \in [0,1]$ with $f(0) < 0$ and $f(1) > 0$.

Consider the switching strategy

$$s(x_i) = \begin{cases} 1, & \text{if } x_i \leq \bar{x}^e \\ 0, & \text{if } x_i > \bar{x}^e \end{cases},$$

when $\bar{x}^e = -e + (1+2e)(1-\bar{u})$.

Assuming all other players use the described strategy $s(x_i)$, we have

$$A(\theta) = \int_{\theta-e}^{\theta+e} s(x_i)dx_i = \begin{cases} 1, & \text{if } \theta \leq \bar{x}^e - e \\ \frac{(\bar{x}^e+e)-\theta}{2e}, & \text{if } \theta \in [\bar{x}^e - e, \bar{x}^e + e] \\ 0, & \text{if } \theta \geq \bar{x}^e + e \end{cases}.$$

The expected payoff from attacking conditional on a private signal $\bar{x}^e$ is

$$\begin{aligned} F^A(\bar{x}^e) &= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e}(b(A^s(\theta)))\mathbf{1}_{A^s(\theta)\geq\theta} - c(A^s(\theta))d\theta\right) \\ &= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\frac{\bar{x}^e+e}{1+2e}}(b(A^s(\theta))) - c(A^s(\theta)) + \int_{\frac{\bar{x}^e+e}{1+2e}}^{\bar{x}^e+e}(0 - c(A^s(\theta)))\right)d\theta \\ &= \int_{1-\bar{u}}^{1}(b(A)-c(A))dA - \int_{0}^{1-\bar{u}}c(A)dA \\ &= 0. \end{aligned}$$

The second equality follows from $A^s(\theta) \geq \theta \Leftrightarrow \frac{(\bar{x}^e+e)-\theta}{2e} \geq \theta \Leftrightarrow \frac{\bar{x}^e+e}{1+2e} \geq \theta$. The third equality is a result of a change of variable with $A^s(\theta) = \left(\frac{(\bar{x}^e+e)-\theta}{2e}\right)$.

Since $A^s(\theta) - \theta$ is strictly decreasing, and $A^s(\theta) - \theta < (>) \ 0$ for sufficiently large (small) θ, there exists a unique θ satisfying $A^s(\theta) - \theta = 0$, denoted by $\hat{\theta}$. We then note that this $\hat{\theta} \in (\bar{x}^e - e, \bar{x}^e + e)$. Otherwise if $\hat{\theta} \leq \bar{x}^e - e$, $F^A(\bar{x}^e) = -\frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e} c(A^s(\theta))d\theta\right) < 0$. Else if $\hat{\theta} \geq \bar{x}^e + e$, then $F^A(\bar{x}^e) > 0$. Both cases result in a contradiction.

Now consider $x_i < \bar{x}^e$.

If $x_i \leq \hat{\theta} - e < \bar{x}^e$, then $F^A(x_i) \geq \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e}(b(A^s(\theta)) - c(A^s(\theta)))d\theta\right) \geq 0$.

Else if $\hat{\theta} - e < x_i \leq \bar{x}^e$, then

$$
\begin{aligned}
F^A(x_i) &= \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e}(b(A^s(\theta))\mathbf{1}_{A^s(\theta)\geq\theta} - c(A^s(\theta)))\right)d\theta \\
&= \frac{1}{2e}\left(\int_{x_i-e}^{\hat{\theta}}(b(A^s(\theta)) - c(A^s(\theta))) + \int_{\hat{\theta}}^{x_i+e}(0 - c(A^s(\theta)))\right)d\theta \\
&\geq \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\hat{\theta}}(b(A^s(\theta)) - c(A^s(\theta))) + \int_{\hat{\theta}}^{\bar{x}^e+e}(0 - c(A^s(\theta)))\right)d\theta \\
&= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e}(b(A^s(\theta)))\mathbf{1}_{A^s(\theta)\geq\theta} - c(A^s(\theta))d\theta\right) \\
&= F^A(\bar{x}^e) = 0.
\end{aligned}
$$

That is, $F^A(x_i) \geq 0$, and hence $s(x_i) = 1$ is sustainable $\forall x_i \leq \bar{x}^e$. Similarly, we can prove that $F^A(x_i) < F^A(\bar{x}^e) = 0$ when $x_i > \bar{x}^e$, and hence $s(x_i) = 0$ is rationalizable $\forall x_i > \bar{x}^e$.

Note: The proof just uses the definition of ρ along with the fact that $b(A) - c(A) \geq 0$ for all $A \in [0, \rho]$. Neither the monotonicity assumption (Assumption 14) nor the single crossing assumption (Assumption 13) is required to hold.

□

I.4 Uniqueness under $\Psi = \{s : \mathbb{R} \to [0,1]; A^s(\theta)$ is monotonic in $\theta\}$

Theorem 21 (Uniqueness under Ψ)**.** *For $(b(\cdot), c(\cdot))$ satisfying Assumption 13 and $e > 0$, the strategy profile s^e described in equation (43) is a unique equilibrium in Ψ.*

Proof. Note that there exists a unique $\hat{\theta}$ such that $A^s(\hat{\theta}) = \hat{\theta}$ due to the monotone assumption of A^s and $A^s(\theta) - \theta < 0 \ (> 0)$ for sufficiently large (small) θ.

case 1: $\rho < 1$

We have that $A^s(\theta) \leq \rho$ for all θ. Otherwise, there exists θ_i with $A^s(\theta_i) > \rho$. By monotonicity of $A(\theta)$, $A^s(\theta) > \rho$ for all $\theta \geq \theta_i$. Therefore, for all $x_i \leq \theta_i - e$,

$$\begin{aligned} F(x_i) &= \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e} (b(A^s(\theta))\mathbf{1}_{A^s(\theta)\geq\theta} - c(A^s(\theta)))d\theta\right) \\ &\leq \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e} (b(A^s(\theta)) - c(A^s(\theta)))d\theta\right) \\ &< \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e} (b(\rho) - c(\rho))\right) d\theta = 0, \end{aligned}$$

where the last strictly inequality follows from the single crossing assumption of $(b(\cdot), c(\cdot))$. That is, $F(x_i) < 0$, and $s(x_i) = 0$ for all $x_i < \theta_i - e$, conflicting with Lemma 19.

There exists θ_0 with $A^s(\theta_0) = \rho$. If $A^s(\theta) < \rho$ for all θ, then, for all $x_i \leq \hat{\theta} - e$,

$$\begin{aligned} F(x_i) &= \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e} (b(A^s(\theta))\mathbf{1}_{A^s(\theta)\geq\theta} - c(A^s(\theta)))d\theta\right) \\ &= \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e} (b(A^s(\theta)) - c(A^s(\theta)))d\theta\right) \\ &> \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e} (b(\rho) - c(\rho))d\theta\right) \\ &= 0. \end{aligned}$$

Again, the last strict inequality follows from the single crossing assumption of $(b(\cdot), c(\cdot))$. That is, $F(x_i) > 0$ for all $x_i \leq \hat{\theta} - e$. We have $\rho > A^s(\hat{\theta} - 2e) = 1 \geq \rho$, which is a contradiction.

Now, let

$$\theta_\rho = \sup\{\theta \in \mathbb{R}; A^s(\theta) = \rho\} \text{ and}$$

$$\bar{x}^e = \sup\{x \in [\hat{\theta} - e, \hat{\theta} + e); F(x) = 0\}.$$

θ_ρ is well-defined because $A^s(\theta_0) = \rho$, $A^s(\hat{\theta} + e) = 0$, and $A^s(\theta)$ is continuous and weakly decreasing.

$\bar{x}^e$ is well-defined because

1. $F(\hat{\theta} - e) = \frac{1}{2e}\left(\int_{\hat{\theta}-2e}^{\hat{\theta}} (b(A^s(\theta)) - c(A^s(\theta)))d\theta\right) \geq b(\rho) - c(\rho) = 0$, where the inequality follows from the single crossing assumption of $(b(\cdot), c(\cdot))$,
2. $F(\hat{\theta} + e) = \frac{1}{2e}\left(\int_{\hat{\theta}}^{\hat{\theta}+2e} (0 - c(A^s(\theta)))d\theta\right) < 0$,
3. $F(x_i)$ is continuous in x_i.

Therefore, there exists $\bar{x} \in [\hat{\theta} - e, \hat{\theta} + e)$ such that $F(\bar{x}) = 0$, and the supremum must exist. We label this supremum $\bar{x}^e$. That is, $\bar{x}^e$ is the rightmost indifference point after which attacking becomes suboptimal.

We claim that $\theta_\rho < \hat{\theta}$. If $\theta_\rho \geq \hat{\theta}$, then for all $x_i > \hat{\theta} - e$,

$$\begin{aligned}
F(x_i) &= \frac{1}{2e}\int_{x_i-e}^{x_i+e}(b(A^s(\theta))\mathbf{1}_{A^s(\theta)\geq\theta} - c(A^s(\theta)))d\theta \\
&= \frac{1}{2e}\left(\int_{x_i-e}^{\hat{\theta}}(b(A^s(\theta)) - c(A^s(\theta)))d\theta + \int_{\hat{\theta}}^{x_i+e}(0 - c(A^s(\theta)))d\theta\right) \\
&\leq \frac{1}{2e}\left(\int_{x_i-e}^{\hat{\theta}}(b(\rho) - c(\rho))d\theta + \int_{\hat{\theta}}^{x_i+e}(0 - c(A^s(\theta)))d\theta\right) \\
&= 0 - \frac{1}{2e}\int_{\hat{\theta}}^{x_i+e} c(A^s(\theta))d\theta) \\
&< 0.
\end{aligned}$$

Again, we use the single crossing assumption of $(b(\cdot), c(\cdot))$ in the third line. We now have that $F(x_i) < 0$ for all $x_i > \hat{\theta} - e$ and $0 = A^s(\hat{\theta}) \geq \rho$, which is a contradiction. So, we must have $\theta_\rho < \hat{\theta}$

It is then straightforward to check the following properties:

1. $F(x_i) < 0$ for all $x_i > \bar{x}^e$
2. $F(\bar{x}^e) = 0$
3. $F(x_i) > 0$ for all $x_i \in (\theta_\rho - e, \bar{x}^e)$
4. $F(x_i) = 0$ for all $x_i \leq \theta_\rho - e$
5. $A(\theta) = \rho$ for all $\theta \leq \theta_\rho$

For $F(x_i) > 0$ for all $x_i \in (\theta_\rho - e, \bar{x}^e)$, it is helpful to show (1) $F(x_i) > 0$ for all $x_i \in (\theta_\rho - e, \hat{\theta} - e)$ and (2) $F(x_i) > 0$ for all $x_i \in [\hat{\theta} - e, \bar{x}^e)$. The first equality follows from $A^s(\theta) < \rho$ for some $\theta \in (\theta_\rho, \hat{\theta}]$, while the second inequality follows directly as $F(\bar{x}^e) = 0$ and $F(x)$ is decreasing over $x \in [\hat{\theta} - e, \hat{\theta} + e)$.

For given $b(\cdot), c(\cdot)$ and e, $\hat{\theta}$ is unique, which leads to the unique $\bar{x}^e$. Consequently, $(A^s(\theta))_{\theta\in\mathbb{R}}$ and $(s(x_i))_{x_i\in\mathbb{R}}$ are uniquely determined. Now, we characterize the unique $\hat{\theta}$ for given $b(\cdot), c(\cdot)$, and e.

Recall that $A^s(\theta) = \frac{1}{2e}\int_{\theta-e}^{\theta+e} s(x_i)dx_i$ and is characterized by

$$A^s(\theta) = \begin{cases} \rho & \theta \leq \theta_\rho \\ \frac{1}{2e}(\bar{x}^e + e - \theta) & \theta_\rho \leq \theta \leq \bar{x}^e + e\ . \\ 0 & \theta \geq \bar{x}^e + e \end{cases} \tag{55}$$

Equation (55) and $A^s(\theta_\rho)=\rho$ imply that $\theta_\rho=\bar{x}^e+e-2e\rho$. From $F(\bar{x}^e)=0$, we have

$$
\begin{aligned}
0 &= \frac{1}{2e}\int_{\bar{x}^e-e}^{\bar{x}^e+e}(b(A^s(\theta))\mathbf{1}_{A^s(\theta)\geq\theta}-c(A^s(\theta)))d\theta \\
&= \frac{1}{2e}\int_{\bar{x}^e-e}^{\bar{x}^e+e-2e\rho}(b(A^s(\theta))-c(A^s(\theta)))d\theta+\frac{1}{2e}\int_{\bar{x}^e+e-2e\rho}^{\hat{\theta}}(b(A^s(\theta))-c(A^s(\theta)))d\theta \\
&\quad +\frac{1}{2e}\int_{\hat{\theta}}^{\bar{x}^e+e}(0-c(A^s(\theta)))d\theta \\
&= \int_{\rho}^{1}(b(\rho)-c(\rho))dA+\int_{\frac{\bar{x}^e+e-\hat{\theta}}{2e}}^{\rho}(b(A)-c(A))dA+\int_{0}^{\frac{\bar{x}^e+e-\hat{\theta}}{2e}}(0-c(A))dA \\
&= \int_{\frac{\bar{x}^e+e-\hat{\theta}}{2e}}^{\rho}(b(A)-c(A))dA+\int_{0}^{\frac{\bar{x}^e+e-\hat{\theta}}{2e}}(0-c(A))dA
\end{aligned}
$$

Consider $G(u)=\int_{\rho-u}^{\rho}(b(A)-c(A))dA+\int_0^{\rho-u}(0-c(A))dA$. Note that $G(0)<0$, $G(\rho)>0$, and $G'(u)=b(\rho-u)>0$. Therefore, there is a unique u satisfying $G(u)=0$. We label this u, $\bar{u}$ and have that $\hat{\theta}=\bar{x}^e+e-2e(\rho-\bar{u})$. From equation (55),

$$\hat{\theta}=A(\hat{\theta})=A(\bar{x}^e+e-2e\rho+2e\bar{u})=\rho-\bar{u}.$$

That is, $\hat{\theta}$ is uniquely determined.

case 2: $\rho=1$

Denote $\bar{u}\in(0,1)$ a unique solution to

$$G(u):=\int_{1-u}^{1}(b(A)-c(A))dA-\int_0^{1-u}c(A)dA=0.$$

Such $\bar{u}$ exists and is unique because $G(u)$ is strictly increasing $\forall u\in[0,1]$ with $G(0)<0$ and $G(1)>0$.

Consider the switching strategy

$$s(x_i)=\begin{cases}1, & \text{if } x_i\leq\bar{x}^e\\ 0, & \text{if } x_i>\bar{x}^e\end{cases},$$

when $\bar{x}^e=-e+(1+2e)(1-\bar{u})$.

Assuming all other players use the described strategy $s(x_i)$, we have

$$A(\theta) = \int_{\theta-e}^{\theta+e} s(x_i)dx_i = \begin{cases} 1, & \text{if } \theta \leq \bar{x}^e - e \\ \frac{(\bar{x}^e+e)-\theta}{2e}, & \text{if } \theta \in [\bar{x}^e - e, \bar{x}^e + e] \\ 0, & \text{if } \theta \geq \bar{x}^e + e \end{cases} .$$

The expected payoff from attacking conditional on a private signal $\bar{x}^e$ is

$$\begin{aligned} F^A(\bar{x}^e) &= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e} (b(A^s(\theta))\mathbf{1}_{A^s(\theta)\geq\theta} - c(A^s(\theta)))d\theta\right) \\ &= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\frac{\bar{x}^e+e}{1+2e}} (b(A^s(\theta)) - c(A^s(\theta)))d\theta\right) + \frac{1}{2e}\left(\int_{\frac{\bar{x}^e+e}{1+2e}}^{\bar{x}^e+e} (0 - c(A^s(\theta)))d\theta\right) \\ &= \int_{1-\bar{u}}^{1} (b(A) - c(A))dA - \int_0^{1-\bar{u}} c(A)dA \\ &= 0. \end{aligned}$$

The second equality follows from $A^s(\theta) \geq \theta \Leftrightarrow \frac{(\bar{x}^e+e)-\theta}{2e} \geq \theta \Leftrightarrow \frac{\bar{x}^e+e}{1+2e} \geq \theta$.

Since $A^s(\theta) - \theta$ is strictly decreasing, and $A^s(\theta) - \theta < (>)\ 0$ for sufficiently large (small) θ, there exists a unique θ satisfying $A^s(\theta) - \theta = 0$, denoted by $\hat{\theta}$. We then note that this $\hat{\theta} \in (\bar{x}^e - e, \bar{x}^e + e)$. Otherwise, if $\hat{\theta} \leq \bar{x}^e - e$, $F^A(\bar{x}^e) = -\frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e} c(A^s(\theta))d\theta\right) < 0$. Similar argument will have $F^A(\bar{x}^e) > 0$. Both cases result in a contradiction.

Consider $x_i < \bar{x}^e$.

If $x_i \leq \hat{\theta} - e < \bar{x}^e$, then

$$\begin{aligned} F^A(x_i) &= \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e} (b(A^s(\theta))\mathbf{1}_{A^s(\theta)\geq\theta} - c(A^s(\theta)))d\theta\right) \\ &= \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e} (b(A^s(\theta)) - c(A^s(\theta)))d\theta\right) \geq 0. \end{aligned}$$

Else if $\hat{\theta} - e < x_i \leq \bar{x}^e$, then

$$\begin{aligned}
F^A(x_i) &= \frac{1}{2e}\left(\int_{x_i-e}^{x_i+e}(b(A^s(\theta))\mathbf{1}_{A^s(\theta)\geq\theta} - c(A^s(\theta)))d\theta\right) \\
&= \frac{1}{2e}\left(\int_{x_i-e}^{\hat{\theta}}(b(A^s(\theta)) - c(A^s(\theta)))d\theta + \int_{\hat{\theta}}^{x_i+e}(0 - c(A^s(\theta)))d\theta\right) \\
&\geq \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\hat{\theta}}(b(A^s(\theta)) - c(A^s(\theta)))d\theta + \int_{\hat{\theta}}^{\bar{x}^e+e}(0 - c(A^s(\theta)))d\theta\right) \\
&= \frac{1}{2e}\left(\int_{\bar{x}^e-e}^{\bar{x}^e+e}(b(A^s(\theta))\mathbf{1}_{A^s(\theta)\geq\theta} - c(A^s(\theta)))d\theta\right) \\
&= F^A(\bar{x}^e) = 0.
\end{aligned}$$

That is, $F^A(x_i) \geq 0$ for all $x_i < \bar{x}^e$, and $s(x_i) = 1$ is sustainable $\forall x_i \leq \bar{x}^e$. Similar, we can show that $F^A(x_i) < F^A(\bar{x}^e) = 0$ when $x_i > \bar{x}^e$, and hence $s(x_i) = 0$ is rationalizable $\forall x_i > \bar{x}^e$.

Note: The proof only requires $b(\cdot)$ and $c(\cdot)$ to satisfy the single crossing assumption from Assumption 13.

□

I.5 Convergence to Monotone Equilibrium

Proposition 22 (Convergence to Monotone Equilibrium)**.** *The strategy s^e defined by equation (43) converges in distribution to s^0 as $e \to 0$. That is, for all $\epsilon > 0$, there exists $\delta > 0$ such that $e \in [0, \delta) \Rightarrow \int_{\theta\in\mathbb{R}} \mid A^{s^e}(\theta) - A^{s^0}(\theta) \mid d\theta < \epsilon$.*

Proof. It is straightforward to check that $A^{s^e}(\theta) = A^{s^0}(\theta) = \rho$ for $\theta < \min(\bar{x}^e + e - 2e\rho, \bar{x}^0)$, $A^{s^e}(\theta) = A^{s^0}(\theta) = 0$ for $\theta > \max(\bar{x}^e + e, \bar{x}^0)$, and $0 \leq A^{s^e}(\theta), A^{s^0}(\theta) \leq \rho$ otherwise. Thus, $\int_{\theta\in\mathbb{R}} \mid A^{s^e}(\theta) - A^{s^0}(\theta) \mid\leq (2e\rho)\rho = 2e\rho^2$, which converges to 0 as $e \to 0$. □

Note: Neither the monotonicity assumption (Assumption 14) nor the single crossing assumption (Assumption 13) is required to hold.

I.6 Monotonicity and Continuity of the Switching Threshold with respect to the Substitutability

Proposition 23 (Monotonicity and Continuity)**.** *For any payoff function $(b(\cdot), c(\cdot))$ and a given noise $e \geq 0$,*

1. *Continuity:* $\bar{x}^e$ *is continuous in* $(b(\cdot), c(\cdot))$ *endowed with the supremum norm. That is,* $\forall \epsilon > 0\ \exists \delta_1 > 0\ \exists \delta_2 > 0$ *such that* $\sup_{A\in[0,1]} \parallel (b_1(A) - c_1(A)) - (b_2(A) - c_2(A)) \parallel < \delta_1$ *and* $\sup_{A\in[0,1]} \parallel c_1(A) - c_2(A) \parallel < \delta_2 \Rightarrow \mid \bar{x}_1^e - \bar{x}_2^e \mid < \epsilon$

2. *Monotonicity:* $\bar{x}^e$ *is monotonically increasing in* $(b(\cdot) - c(\cdot), -c(\cdot))$ *under the pointwise dominance order. That is, if* $b_1(A) - c_1(A) \geq b_2(A) - c_2(A)$ *and* $c_1(A) \leq c_2(A)$ *for all* $A \in [0,1]$*, then* $\bar{x}_1^e \geq \bar{x}_2^e$*.*

Proof. Consider $e \geq 0$ and a payoff function $b_1(\cdot)$ and $c_1(\cdot)$, we show the following:

1) Continuity: For any $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that for any payoff function $(b_2(\cdot), c_2(\cdot))$ satisfying $\sup_{A\in[0,1]} \parallel (b_1(A) - c_1(A)) - (b_2(A) - c_2(A)) \parallel < \delta_1$ and $\sup_{A\in[0,1]} \parallel b_1(A) - b_2(A) \parallel < \delta_2$, we have $\mid \bar{x}_1^e - \bar{x}_2^e \mid < \epsilon$ when $\bar{x}_1^e$ and $\bar{x}_2^e$ are as defined in equation (44) corresponding to $(b_1(\cdot), c_1(\cdot))$ and $(b_2(\cdot), c_2(\cdot))$, respectively.

Fix $\epsilon > 0$, define the associated degree of substitutability ρ_1 and ρ_2 as in equation (16).

If $b_1(A) - c_1(A) > 0$ for all $A \in [0,1]$, we have $\rho_1 = 1$, and there exists M_1 such that $b_1(A) - c_1(A) \geq M_1 > 0$ (otherwise by the continuity of $b_1(\cdot) - c_1(\cdot)$. there will be $\tilde{A} \in [0,1]$ such that $b_1(\tilde{A}) - c_1(\tilde{A}) = 0$, which is a contradiction). Pick $\delta_1 = \frac{M_1}{2}$, we have that $b_2(A) - c_2(A) = b_1(A) - c_1(A) + [(b_2(A) - c_2(A)) - (b_1(A) - c_1(A))] \geq b_1(A) - c_1(A) - \mid [(b_2(A) - c_2(A)) - (b_1(A) - c_1(A))] \mid \geq b_1(A) - c_1(A) - \frac{M_1}{2} \geq b_1(A) - c_1(A) - (\frac{b_1(A)-c_1(A)}{2}) = \frac{b_1(A)-c_1(A)}{2} > 0$. That is, $\rho_2 = 1$. In this case, $\rho_1 - \rho_2 = 0$.

Else, there are $\rho_1, \rho_2 \in [0,1]$ such that $b_1(\rho_1) - c_1(\rho_1) = 0$ and $b - 2(\rho_2) - c_2(\rho_2) = 0$.

$$\begin{aligned}
&\sup_{A\in[0,1]} \parallel (b_1(A) - c_1(A)) - (b_2(A) - c_2(A)) \parallel < \delta_1 \\
&\Rightarrow \mid (b_1(\rho_1) - c_1(\rho_1)) - (b_2(\rho_1) - c_2(\rho_1)) \mid < \delta_1 \\
&\quad \text{and } \mid (b_1(\rho_2) - c_1(\rho_2)) - (b_2(\rho_2) - c_2(\rho_2)) \mid < \delta_1
\end{aligned}$$

By the Mean Value Theorem, there is ρ^* between ρ_1 and ρ_2

$$(b_1(\rho_1) - c_1(\rho_1)) - (b_1(\rho_2) - c_1(\rho_2)) = (b_1'(\rho^*) - c_1'(\rho^*))(\rho_1 - \rho_2).$$

Thus,

$$\mid \rho_1 - \rho_2 \mid = \frac{\mid (b_1(\rho_1) - c_1(\rho_1)) - (b_1(\rho_2) - c_1(\rho_2)) \mid}{\mid b_1'(\rho^*) \mid}.$$

For any $\epsilon > 0$, we can select $\delta_1 = \frac{|b_1'(\rho^*)|}{2}$ such that $\mid \rho_1 - \rho_2 \mid < \epsilon$.

Next, from the definition of $\bar{u}_1$ and $\bar{u}_2$, we have that

$$\int_{\rho_1-\bar{u}_1}^{\rho_1}(b_1(A)-c_1(A))dA = \int_0^{\rho_1-\bar{u}_1} c_1(A)dA, \tag{56}$$

and

$$\int_{\rho_2-\bar{u}_2}^{\rho_2}(b_2(A)-c_2(A))dA = \int_0^{\rho_2-\bar{u}_2} c_2(A)dA. \tag{57}$$

Without loss of generality, assume that $\rho_1 > \rho_2$. We can ensure that the selected δ is sufficiently small that $\rho_2 > \rho_1 - \bar{u}_1$. Subtracting equation (57) from equation (56), we have

$$\begin{aligned}
&\int_{\rho_2}^{\rho_1}(b_1(A)-c_1(A))dA + \int_{\max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}^{\rho_2}(b_2(A)+c_2(A)-b_1(A)-c_1(A))dA + \\
&\quad (-1)^{i+1}\int_{\min(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}^{\max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}(b_i(A)-c_i(A))dA \\
&= \int_0^{\min(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}(c_1(A)-c_2(A))dA + (-1)^{i+1}\int_{\min(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}^{\max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)} c_i(A)dA,
\end{aligned}$$

where i is such that $\min(\rho_1-\bar{u}_1, \rho_2-\bar{u}_2) = \rho_i - \bar{u}_i$.

Let $M_2 = \inf_{A\in[0,1]} b_i(A)$. The triangle's inequality implies that

$$\begin{aligned}
\mid (\rho_1-\bar{u}_1)-(\rho_2-\bar{u}_2) \mid M_2 &\leq \mid (-1)^{i+1}\int_{\min(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}^{\max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)} b_i(A)dA \mid \\
&\leq \mid \int_0^{min(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}(c_1(A)-c_2(A))dA \mid \\
&\quad + \mid \int_{\max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)}^{\rho_2}(b_2(A)-c_2(A)-b_1(A)+c_1(A))dA \mid \\
&\quad + \mid \int_{\rho_2}^{\rho_1}(b_1(A)-c_1(A))dA \mid \\
&\leq \delta_2 \min(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2) + \delta_1(\rho_2 - \max(\rho_1-\bar{u}_1,\rho_2-\bar{u}_2)) + \\
&\qquad \max_{A\in[\rho_2,\rho_1]} \mid b_1(A)-c_1(A) \mid (\rho_1-\rho_2).
\end{aligned}$$

For $\epsilon > 0$, we can always pick $\delta_1 > 0$ and $\delta_2 > 0$ small enough such that $\mid \bar{x}_1^e - \bar{x}_2^e \mid = (2e+1) \mid (\rho_1-\bar{u}_1)-(\rho_2-\bar{u}_2) \mid < \epsilon$.

2) Monotonicity: We want to show that $b_1(\cdot)-c_1(\cdot) \geq b_2(\cdot)-c_2(\cdot)$ and $c_1(\cdot) \leq c_2(\cdot) \Rightarrow \bar{x}_1^e \geq \bar{x}_2^e$.

Assume $b_1(\cdot)-c_1(\cdot) \geq b_2(\cdot)-c_2(\cdot)$, we have that $\rho_1 \geq \rho_2$. It is left to show that $\rho_1-\bar{u}_1 \geq \rho_2-\bar{u}_2$. Suppose otherwise, i.e. $\rho_1-\bar{u}_1 < \rho_2-\bar{u}_2$. This implies that $\rho_1 \geq \rho_2 \geq$

$\rho_2 - \bar{u}_2 > \rho_1 - \bar{u}_1$. Subtracting equation (57) from (56), we have

$$\int_{\rho_1}^{\rho_2}(b_1(A) - c_1(A))dA + \int_{\rho_2-\bar{u}_2}^{\rho_2}(b_1(A) - c_1(A) - b_2(A) - c_2(A))dA + \\ \int_{\rho_2-\bar{u}_2}^{\rho_1-\bar{u}_1}(b_1(A) - c_1(A))dA = \int_0^{\rho_1-\bar{u}_1}(c_1(A) - c_2(A))dA - \int_{\rho_1-\bar{u}_1}^{\rho_2-\bar{u}_2} c_2(A)dA \quad (58)$$

Now, we note that the RHS of equation (58) is negative, while the LHS of equation (58) is positive due to the ordering of payoffs along with the definition of ρ. We arrive at the contradiction. Therefore, $\rho_1 - \bar{u}_1 \geq \rho_2 - \bar{u}_2$ which implies $\bar{x}_1^e \geq \bar{x}_2^e$ as desired.

Note: The proof just uses the definition of ρ along with the fact that $b(A) - c(A) \geq 0$ for all $A \in [0, \rho]$. Neither the monotonicity assumption (Assumption 14) nor the single crossing assumption (Assumption 13) is required to hold.

□

I.7 External Attacking Pressure

Proposition 24 (External Attacking Pressure)**.** *External attacking pressure moves the switching threshold below which the regime switches to the right. That is, $\frac{\partial\hat{\theta}}{\partial X} > 0$, when $\hat{\theta}$ is the fundamental threshold.*

Proof. Suppose there exists an external selling pressure $X \geq 0$. The individual payoff is now modified to

$$u_i(a_i, A, \theta) = a_i \left((b(A + X))\mathbf{1}_{A+X\geq\theta} - c(A + X) \right).$$

It is straightforward to show that the following strategy described a Nash equilibrium of regime-change games with external selling pressure X.

$$s^e(x_i) = \begin{cases} 0 & ; x_i \geq \bar{x}^e \\ 1 & ; \bar{x}^e - 2ek - 2e(\rho - X) \leq x_i < \bar{x}^e - 2ek, \quad k = 0, 1, 2, \ldots \\ 0 & ; \bar{x}^e - 2e(k+1) \leq x_i < \bar{x}^e - 2ek - 2e(\rho - X), \quad k = 0, 1, 2, \ldots \end{cases}, \quad (59)$$

when $\bar{x}^e$ is given by

$$\bar{x}^e = -e + (1 + 2e)(\hat{\theta}_X - X) + X, \quad (60)$$

and $\hat{\theta}_X$ is the unique value $\theta \in (X, \rho)$ satisfying

$$\int_\theta^\rho (b(A) - c(A))dA = \int_X^\theta c(A)dA. \quad (61)$$

Let $G(\theta) = \int_\theta^\rho (b(A) - c(A))dA - \int_X^\theta c(A)dA$. Note that $G(X) > 0$, $G(\rho) < 0$, and $G'(\theta) = -(b(\theta)) < 0$. Therefore, there exists a unique $\hat{\theta}_X$ such that $G(\hat{\theta}_X) = 0$ holds.

Note that $A^s(\theta) = \begin{cases} \rho - X & \text{for } \theta \leq \bar{x}^e - 2e(\rho - X) + e \\ \frac{\bar{x}^e + e - \theta}{2e} & \text{for } \theta \in [\bar{x}^e - 2e(\rho - X) + e, \bar{x}^e + e] \\ 0 & \text{for } \theta \geq \bar{x}^e + e \end{cases}$, and $A^s(\hat{\theta}_X) + X = \hat{\theta}_X$.

Note that $\hat{\theta}_X$ is the fundamental threshold below which the regime switches.

We are particularly interested in how X affects this fundamental threshold $\hat{\theta}_X$. Taking the derivative of equation (61) with respect to X using Leibniz's rule, we have

$$-(b(\hat{\theta}_X) - c(\hat{\theta}_X))\frac{\partial \hat{\theta}_X}{\partial X} = \frac{\partial \hat{\theta}_X}{\partial X} c(\hat{\theta}_X) - c(X)$$
$$\frac{\partial \hat{\theta}_X}{\partial X} = \frac{c(X)}{b(\hat{\theta}_X)} > 0.$$

That is, an external attacking pressure always moves the fundamental switching threshold to the right.

Note: The proof relies neither on the single crossing assumption (Assumption 13). However, recall that we have the (local) uniqueness theorem only this assumption holds. Without the single crossing assumption, we can still identify one (of potentially many) equilibrium strategy in the private information case. The proof above shows that external attacking pressure moves the threshold associated with this equilibrium strategy to the right.

□

I.8 Reevaluating Policy Implications: The Effect of Quota

Proposition 25 (The Effect of Quota)**.** *Consider any payoff functions $(b(\cdot), c(\cdot))$ satisfying the monotonicity assumption (Assumption 14). let ρ be as defined in equation (16), and $\hat{\theta}$ and $\hat{\theta}_Q$ be fundamental thresholds below which regime switches when there is no quota and when the quota is set at $Q \in [0, 1]$, respectively.*

1. *Without crowding out, $b(A) \equiv \bar{b}$ and $c(A) \equiv \bar{c}$ for $\bar{b}, \bar{c} \in \mathbb{R}_+$ with $\bar{b} > \bar{c}$, an imposed quota $Q \in [0, 1]$ never expands the fundamental domain where the regime switches. That is, $\hat{\theta}_Q \leq \hat{\theta}$.*

2. *When the crowding out is present, an imposed quota can expand the fundamental domain where the regime switches.*

- *When the quota is quite restrictive, $Q \le \hat{\theta}$, the fundamental switching threshold is lower, i.e. $\hat{\theta}_Q \le \hat{\theta}$.*
- *For an intermediate level of quota, $Q \in (\hat{\theta}, \rho)$, there exists a payoff function $(b(\cdot), c(\cdot))$ such that the fundamental switching threshold is higher, i.e. $\hat{\theta} < \hat{\theta}_Q$.*
- *A quota that is bigger than the degree of substitutability, $Q \ge \rho$, does not affect the fundamental switching threshold. That is, $\hat{\theta}_Q = \hat{\theta}$*

Proof. For a given payoff function $(b(\cdot), c(\cdot))$. Let $\hat{\theta}$ and $\hat{\theta}_Q$ be fundamental thresholds when there is no imposed quota and when a quota is set at Q respectively.

case 1: $Q \le \hat{\theta}$

Let s_Q be an equilibriums strategy when the quota is present. Observing $x_i \ge Q + e$, speculators know that the fundamental $\theta \ge Q$, and attack is never successful. $F^{s_Q}(x_i) < 0$. Therefore, $s_Q(x_i) = 0$ for all $x_i \ge Q + e$. Let $\bar{x}^e_Q = \sup\{x_i : F^{s_Q}(x_i) \ge 0\}$, we have that $\bar{x}^e_Q < Q + e$. In addition, we have that $A^{s_Q}(\theta) \le Q$ for all $\theta \ge \bar{x}_Q + e - 2eQ$.

At the fundamental threshold $\hat{\theta}_Q$, we must have that $A^{s_Q}(\hat{\theta}_Q) = \hat{\theta}_Q$. We have the following:

$$\begin{aligned}\hat{\theta}_Q &= A^{s_Q}(\hat{\theta}_Q)\\ &\ge Q\\ &\ge A^{s_Q}(\theta) \text{ for all } \theta \ge \bar{x}^e_Q + e - 2eQ.\end{aligned}$$

We still restrict our search of strategy profiles over all those with monotone aggregate action. From $A^{s_Q}(\hat{\theta}_Q) \ge A^{s_Q}(\theta)$ for all $\theta \ge \bar{x}^e_Q + e - 2eQ$, we have that $\hat{\theta}_Q \le \bar{x}^e_Q + e - 2eQ < Q + 2e(1-Q)$. In the limit of precise private signals, we have $\hat{\theta}_Q \le Q \le \hat{\theta}$. That is, $\hat{\theta}_Q \le \hat{\theta}$ for all $Q \le \hat{\theta}$.

When the quota is restrictive, the fundamental switching threshold is always lower than the threshold when the quota is not present.

case 2: $Q > \hat{\theta}$

$b(A) \equiv \bar{b}$ **and** $c(A) \equiv \bar{c}$ **for some** $\bar{b}, \bar{c} \in \mathbb{R}_+$ **with** $\bar{b} > \bar{c}$

First note that $\bar{b} - \bar{c} > 0$ implies that $\frac{Q}{A}(\bar{b} - \bar{c}) > 0$ for all $A \in [0,1]$. We know from Theorem 20 that $\hat{\theta}$ and $\hat{\theta}_Q$ are defined respectively as below:

$$\int_{\hat{\theta}}^{1} (\bar{b} - \bar{c}) dA = \int_0^{\hat{\theta}} \bar{c} dA$$
$$\int_Q^1 \frac{Q}{A}(\bar{b} - \bar{c}) dA + \int_{\hat{\theta}_Q}^{Q} (\bar{b} - \bar{c}) dA = \int_0^{\hat{\theta}_Q} \bar{c} dA.$$

Define $M(\theta) = \int_Q^1 \frac{Q}{A}(\bar{b}-\bar{c})dA + \int_\theta^Q (\bar{b}-\bar{c})dA - \int_0^\theta \bar{c}dA$. We have that $\frac{\partial M(\theta)}{\partial\theta} < 0$ and $F(\hat{\theta}) < 0$, therefore we must have that $\hat{\theta}_Q < \hat{\theta}$.

Ambiguity Effects of Quota

There exists a payoff function $(b(\cdot), c(\cdot))$ such that $\frac{Q}{A}(b(Q)-c(Q)) \geq b(A)-c(A)$ for all $A \in [Q,1]$, and the inequality is strict over a non-zero measure interval that is a subset of $[Q,1]$.

If $b(A)-c(A) \geq 0$ for all $A \in [0,1]$, $\rho = 1$ and $\hat{\theta}$ is such that

$$\int_{\hat{\theta}}^1 (b(A)-c(A))dA = \int_0^{\hat{\theta}} c(A)dA.$$

Define $M(\theta) = \int_Q^1 \frac{Q}{A}(b(A)-c(A))dA + \int_\theta^Q (b(A)-c(A))dA - \int_0^\theta c(A)dA$. Again, we have $\frac{\partial M(\theta)}{\partial\theta} < 0$. However, here $M(\hat{\theta}) > \int_{\hat{\theta}}^1 (b(A)-c(A))dA - \int_0^{\hat{\theta}} c(A)dA = 0$. Therefore, $\hat{\theta}_Q$ such that $M(\hat{\theta}_Q) = 0$ must be greater than $\hat{\theta}$. That is, $\hat{\theta} < \hat{\theta}_Q$.

If $b(1)-c(A) < 0$, then $\rho < 1$. Now $\hat{\theta}$ is defined from Theorem 20 as below:

$$\int_{\hat{\theta}}^\rho (b(A)-c(A))dA = \int_0^{\hat{\theta}} c(A)dA. \tag{62}$$

There are 2 cases.

1. $Q < \rho$ or $b(Q)-c(Q) > 0$

   Define $M(\theta) = \int_Q^1 \frac{Q}{A}(b(A)-c(A))dA + \int_\theta^Q (b(A)-c(A))dA - \int_0^\theta c(A)dA$. Again, we have $\frac{\partial M(\theta)}{\partial\theta} < 0$. However, here $M(\hat{\theta}) > \int_{\hat{\theta}}^\rho \frac{Q}{A}(b(A)-c(A))dA - \int_0^{\hat{\theta}} c(A)dA \geq \int_{\hat{\theta}}^\rho (b(A)-c(A))dA - \int_0^{\hat{\theta}} c(A)dA = 0$. Therefore, $\hat{\theta}_Q$ such that $M(\hat{\theta}_Q) = 0$ must be greater than $\hat{\theta}$. That is, $\hat{\theta} < \hat{\theta}_Q$.

2. $Q \geq \rho$ or $b(Q)-c(Q) \leq 0$

   The fundamental threshold $\hat{\theta}_Q$ is then defined by the following equation:

$$\int_{\hat{\theta}Q}^1 (b(A)-c(A))dA = \int_0^{\hat{\theta}} c(A)dA,$$

   which is exactly the same as equation (62). That is, $\hat{\theta}_Q = \hat{\theta}$.

$\square$

References

George-Marios Angeletos, Christian Hellwig, and Alessandro Pavan. Dynamic global games of regime change: Learning, multiplicity, and the timing of attacks. *Econometrica*, 75(3):711–756, 2007.

Susan Athey. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica*, 69(4):861–889, 2001.

Guillermo A Calvo. Servicing the public debt: The role of expectations. *The American Economic Review*, pages 647–661, 1988.

Hans Carlsson and Eric Van Damme. Global games and equilibrium selection. *Econometrica: Journal of the Econometric Society*, pages 989–1018, 1993.

Harold L Cole and Timothy J Kehoe. Self-fulfilling debt crises. *The Review of Economic Studies*, 67(1):91–116, 2000.

Giancarlo Corsetti, Paolo Pesenti, and Nouriel Roubini. The role of large players in currency crises. In *Preventing Currency Crises in Emerging Markets*, pages 197–268. University of Chicago Press, 2002.

Giancarlo Corsetti, Amil Dasgupta, Stephen Morris, and Hyun Song Shin. Does one Soros make a difference? A theory of currency crises with large and small traders. *The Review of Economic Studies*, 71(1):87–113, 2004.

Eduardo Davila and Ansgar Walther. Does size matter? bailouts with large and small banks. Technical report, National Bureau of Economic Research, 2017.

Douglas W Diamond and Philip H Dybvig. Bank runs, deposit insurance, and liquidity. *Journal of political economy*, 91(3):401–419, 1983.

Chris Edmond. Information manipulation, coordination, and regime change. *Review of Economic Studies*, 80(4):1422–1458, 2013.

Emmanuel Farhi and Jean Tirole. Collective moral hazard, maturity mismatch, and systemic bailouts. *American Economic Review*, 102(1):60–93, 2012.

Xavier Gabaix and Matteo Maggiori. International liquidity and exchange rate dynamics. *The Quarterly Journal of Economics*, 130(3):1369–1420, 2015.

Itay Goldstein and Ady Pauzner. Demand–deposit contracts and the probability of bank runs. *the Journal of Finance*, 60(3):1293–1327, 2005.

Zhiguo He and Wei Xiong. Dynamic debt runs. *Review of Financial Studies*, 25(6):1799–1843, 2012.

Zhiguo He, Arvind Krishnamurthy, Konstantin Milbradt, et al. A model of the reserve asset. Technical report, 2015.

Eric J Hoffmann and Tarun Sabarwal. A global game with strategic substitutes and complements: Comment. *Games and Economic Behavior*, 94:188–190, 2015.

Larry Karp, In Ho Lee, and Robin Mason. A global game with strategic substitutes and complements. *Games and Economic Behavior*, 60(1):155–175, 2007.

Paul Milgrom and Chris Shannon. Monotone comparative statics. *Econometrica: Journal of the Econometric Society*, pages 157–180, 1994.

Stephen Morris and Hyun Song Shin. Unique equilibrium in a model of self-fulfilling currency attacks. *American Economic Review*, pages 587–597, 1998.

Stephen Morris and Hyun Song Shin. Rethinking multiple equilibria in macroeconomic modeling. *NBER macroeconomics Annual*, 15:139–161, 2000.

Stephen Morris and Hyun Song Shin. Global games: Theory and applications. In *Advances in Economics and Econometrics (Proceedings of the Eighth World Congress of the Econometric Society)*, volume 1, pages 56–114. Cambridge University Press, 2003.

Stephen Morris and Hyun Song Shin. Coordination risk and the price of debt. *European Economic Review*, 48(1):133–153, 2004a.

Stephen Morris and Hyun Song Shin. Liquidity black holes. *Review of Finance*, 8(1):1–18, 2004b.

Stephen Morris and Hyun Song Shin. Coordinating expectations: Global games with strategic substitutes. Technical report, mimeo, 2009.

Kevin M Murphy, Andrei Shleifer, and Robert W Vishny. Industrialization and the big push. *Journal of political economy*, 97(5):1003–1026, 1989.

Maurice Obstfeld. Models of currency crises with self-fulfilling features. *European economic review*, 40(3):1037–1047, 1996.

Jean-Charles Rochet and Xavier Vives. Coordination failures and the lender of last resort: was bagehot right after all? *Journal of the European Economic Association*, 2(6):1116–1147, 2004.

Jonathan Weinstein and Muhamet Yildiz. A structure theorem for rationalizability with application to robust predictions of refinements. *Econometrica*, 75(2):365–400, 2007.