The Equilibrium Dynamics of Liquidity and Illiquid Asset Prices*

Adrian Buss Bernard Dumas

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Abstract

When purchasing a security an investor needs not only have in mind the cash flows that the security will pay into the indefinite future, he/she must also anticipate his/her desire and ability to resell the security in the marketplace at a later point in time. In this paper, we show that the endogenous stochastic process of the liquidity of securities is as important to investment and valuation as is the exogenous stochastic process of their future cash flows.

For that purpose, we develop a general-equilibrium model with heterogeneous agents that have an every day motive to trade.

Our method delivers the optimal, market-clearing moves of each investor and the resulting ticker and transactions prices in the presence of transaction costs. We use it to show the effect of transactions costs on asset prices, on deviations from the classic consumption CAPM and on the time path of transactions prices and trades, including their total and quadratic variations. We also show that transactions costs can explain some of the empirical asset-pricing anomalies.

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*Buss is with Goethe University, Frankfurt (buss@finance.uni-frankfurt.de) and Dumas is with INSEAD, NBER and CEPR (bernard.dumas@insead.edu). Work on this topic was initiated while Dumas was at the University of Lausanne. The authors are grateful for useful discussions to Yakov Amihud, Bruno Biais, Massimiliano Croce, Magnus Dahlquist, Xi Dong, Stefano Giglio, Francisco Gomes, Harald Hau, Terrence Hendershott, Julien Hugonnier, Andrew Karolyi, Alexander Michaelides, Pascal Maenhout, Lubos Pastor, Tarun Ramadorai, Olivier Scaillet, Norman Schürhoff, Raman Uppal, Grigory Vilkov, Ingrid Werner and participants at workshops given at the Amsterdam Duisenberg School of Finance, INSEAD, the CEPR's European Summer Symposium in Financial Markets at Gerzensee, the University of Cyprus and the University of Lausanne.

When purchasing a security an investor needs not only have in mind the cash flows that the security will pay into the indefinite future, he/she must also anticipate his/her desire and ability to resell the security in the marketplace at a later point in time. In this paper, we show that the endogenous stochastic process of the liquidity of securities is as important to investment and valuation as is the exogenous stochastic process of their future cash flows.

At any given time, an asset is more or less liquid as a function of three conceivable mechanisms and their fluctuating impact, taken in isolation or combined. The first mechanism is the fear of default of the counterparty to the trade. Trade is obviously hampered by the fear that contracts will not be abided by. The second mechanism is informed trading (asymmetric information) as in the market for "lemons" (Akerlof (1970)). Bhattacharya and Spiegel (1998) have shown the way in which the lemon problem can cause markets to close down and the Microstructure literature stemming from Glosten and Milgrom (1985) and Kyle (1985) has shown that informed trading indirectly generates transactions costs. The third mechanism, which we examine here, is the presence of physical transactions costs.[1]

Access to a financial market is a service that investors make available to each other. The production of this service is achieved by means of a transactions technology that requires some physical input, such as labor, information technology etc., and is, therefore, available at a cost. The physical costs incurred in operating a market are as central to our understanding of financial-market equilibrium as is the production function or the cost function to our understanding of the equilibrium in the market for other goods and services. Given the presence of that cost, an investor may decide not to trade, thereby preventing other investors from trading with him/her, which is an additional endogenous, stochastic and perhaps quantitatively more important consequence of the cost. As a way of providing a simple model, we assume that the trading of a security or the processing of an order entails a physical deadweight cost that is proportional to the number of shares traded and that is paid by both the buyer and the seller.

In the real world, investors do not trade with each other. They trade through intermediaries called brokers, who incur physical costs and charge a fee that is close to being proportional to the value of the shares traded, not to the number of shares traded. This service charge aims to cover the actual physical cost of trading plus a profit. This paper is not about the pricing policy of brokers. We bypass brokers and let the investors incur the physical cost directly.

To our knowledge, the functional form of the physical cost has not been documented very well. A cost proportional to the number of shares seems like a reasonable starting point, although it is clear that there must also exist sizable fixed costs, which we do not consider here.[2]

Our goal is to study, in terms both of price and volume, the dynamics of

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[1] On the various possible determinants of liquidity, see the synthesis paper of Vayanos and Wang (2009).

[2] The equilibrium with other costs, such as holding costs and participation costs, has been investigated by Tuckman and Vila (2010), Huang and Wang (2010) and Peress (2005).

a financial-market equilibrium that we can expect to observe when there are frictions and when investors have an every-day motive for trading, such as shocks to their endowments, that is separate from the long-term need to trade for lifetime planning purposes. Actually, we assume long-lived investors who trade because they have differing risk aversions while they have access only to a menu of linear assets, which, without transactions costs, would be sufficient to make the market dynamically complete. Dynamic completeness is, of course, killed by the presence of transactions costs. The imbalance of the portfolios investors have to hold because of transactions costs act as an inventory cost.

Our paper is related to the existing studies of portfolio choice under transactions costs such as Constantinides (1976a, 1976b, 1986), Davis and Norman (1990), Dumas and Luciano (1991), Edirsinghe, Naik and Uppal (1993), Gennotte and Jung (1994), Shreve and Soner (1994), Leland (2000), Nazareth (2002), Bouchard (2002), Obizhaeva and Wang (2005), Liu and Lowenstein (2002), Jang, Koo, Liu and Lowenstein (2007) and Gerhold, Guasoni, Muhle-Karbe and Schachermayer (2011) among others. As was noted by Dumas and Luciano, these papers suffer from a logical quasi-inconsistency. Not only do they assume an exogenous process for securities returns, as do all portfolio optimization papers, but they do so in a way that is incompatible with the portfolio policy that is produced by the optimization. The portfolio strategy is of a type that recognizes the existence of a "no-trade" region. Yet, it is assumed that prices continue to be quoted and trades remain available in the marketplace.[3] Obviously, the assumption must be made that some traders, other than the one whose portfolio is being optimized, do not incur costs. In the present paper, we assume that all investors face the cost of trading.

In one interpretation, the inventory of securities held by each investor can be viewed as a state variable in the dynamics of our equilibrium, a feature that is shared with the inventory-management model of a broker that has been pioneered by Ho and Stoll (1980, 1983), which is one of the main pillars of the Microstructure literature. In their work, however, Ho and Stoll take the arrival of orders to the broker as an exogenous process.[4] Here, we fully endogenize each investor's decision to trade.

The papers of Vayanos (1998), Vayanos and Vila (1999) and Lo, Mamaysky and Wang (2004) are direct ancestors of the present one, in that they have also postulated a physical cost of transacting and exhibited the resulting equilibrium behavior.[5] However, in Vayanos (1998) and Vayanos and Vila (1999), an

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[3] Constantinides (1986) in his pioneering paper on portfolio choice under transactions costs attempted to draw some conclusions concerning equilibrium. Assuming that returns were independently, identically distributed (IID) over time, he claimed that the expected return required by an investor to hold a security was affected very little by transactions costs. Liu and Lowenstein (2002), Jang, Koo, Liu and Lowenstein (2007) and Dumas an Puopolo (2010) have shown that this is generally not true under non IID returns. The possiblity of falling in a "no-trade" region is obviously a massive violation of the IID assumption.

[4] Recently, Rosu (2009) has developed a model in the same vein in which, however, the brokers interact with each other in a non competive way.

[5] Another predecessor is Milne and Neave (2003), which, however, contains few quantitative results.

investor's only motive to trade is the fact that he has a finite lifetime. Transactions costs induce him to trade twice in his life: when young, he buys some securities that he can resell in order to be able to live during his old age. Here, we introduce a higher-frequency motive to trade. In the paper of Lo, Mamaysky and Wang (2004), costs of trading are fixed costs, all traders have the same negative exponential utility function, individual investors' endowments provide the motive to trade (as in our paper) but aggregate endowment is not stochastic. In our current paper, costs are proportional, utility is a power utility that differs across traders and endowments are free to follow an arbitrary stochastic process. To our knowledge, ours is the first paper to reach that goal.

One can capture liquidity considerations by means of an explicit cost, as we do here, or by a constraint. Holmström and Tirole (2001) study a financial-market equilibrium in which investors face an exogenous constraint on borrowing.[6] When they hit their constraint, investors are said to be "liquidity constrained". Gromb and Vayanos (2002) and Brunnermeier and Pedersen (2008) study situations in which the amount of arbitrage capital is constrained.[7] It would be necessary to present some microfoundations for the constraint. A constraint on borrowing would best be justified by the risk of default on the loan. Equilibrium with default is an important but separate topic of research.

As far as the solution method is concerned, our analysis is closely related, in ways we explain below, to "the dual method" used by Jouini and Kallal (1995), Cvitanic and Karatzas (1996), Kallsen and Muhle-Karbe (2008) and Deelstra, Pham and Touzi (2002) among others.

In computing an equilibrium, one has a choice between a "recursive" method, which solves by backward induction over time, and a "global" method, which solves for all optimality conditions and market-clearing conditions of all states of nature and points in time simultaneously.[8] The global method, often implemented in the form of a homotopy, is limited in terms of the number of periods it can handle. Here, we resort to a recursive technique, which requires the choice of state variables – both exogenous and endogenous – that track the state of the economy. Dumas and Lyasoff (2010) have proposed an efficient method to calculate incomplete-market equilibria recursively with a dual approach, which utilizes state prices as endogenous state variables. We use the same method here. A crucial advantage of using dual variables as state variables to handle proportional-transactions costs problems is that the additional state variables thus introduced evolve on a fixed domain, namely the interval set by the per share cost of buying and the cost of selling (with opposite signs), whereas primal variables, such as portfolio choices evolve over a domain that has free-floating

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[6] As is apparent below, the cost and the constraint approches are somewhat similar but are probably not equivalent to each other. As we show, transaction costs give rise to shadow prices of potentially being unable to trade that are specific to each asset and each investor, whereas a constraint gives rise to a dual variable that is specific to each investor only.

[7] Distant antecedents of this idea in the macroeconomic literature can be found in the form of Clower and Bushaw (1954) constraints, which required a household to hold some money balance, as opposed to being able to borrow, when it wanted to consume, as well as the "cash-in-advance" model of Lucas (1982).

[8] For an implementation of the global solution, see Herings and Schmedders (2006).

barriers, to be determined. The dual problem, for that reason, is more convenient.

Empirical work on equilibria with transactions costs has been couched in terms of a CAPM that recognizes a number of risk factors. Pástor and Stambaugh (2003) and Acharya and Pedersen (2005) have recognized two or more risk factors, one of which is the market return (as in the classic CAPM) or aggregate consumption (as in the consumption-CAPM), and the others are meant to capture stochastic fluctuations in the degree of liquidity of the market, either taken as a whole or individually for each security. Liquidity fluctuations are proxied by fluctuations in volume or in the responsiveness of price to the order flow. The papers cited confirm that there exist in the marketplace significant risk premia related to these factors. Our model also identifies additional risk factors for the investors' willingness to trade, in the form of shadow prices. However, there is one such per investor and they are not directly observable. We use our model to ascertain to what extent proxies used in the empirical literature are to any degree related to these shadow prices.

Indirect evidence on the existence of frictions in the market is provided by empirical anomalies or deviations from some anticipated properties of securities prices. In their recent survey, Gromb and Vayanos (2010) highlight the following anomalies: (a) short-run momentum, the tendency of an asset's recent performance to continue into the near future; (b) long-run reversal, the tendency of performance measured over longer horizons to revert; (c) the value effect, the tendency of an asset's ratio of price to accounting measures of value to predict negatively future returns; (d) the high volatility of asset prices relative to measures of discounted future payoff streams; and (e) post-earnings announcement drift, the tendency of stocks' earning surprises to predict positively future returns. We comment below on the ability of our model to provide explanations of these anomalies.[9] [10]

After writing down our model and specifying the solution method (Section 1), we focus our work on two main questions. First, we ask in Section 2 whether equilibrium securities prices conform to the famous dictum of Amihud and Mendelson (1986a), which says that they are reduced by the present value of transactions costs. In Section 3, we examine the behavior of the market over time, asking, for instance, to what degree price changes and transactions volume are related to each other and what effects transactions costs have on the point process of transaction prices. In Section 4, we quantify the additional premia created by transactions costs, which are deviations from the consump-

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[9] Transactions costs also constitute a "limit to arbitrage" and offer a potential explanation of the observed fact that sometimes securities that are closely related to each other do not trade in the proper price relationship. For these deviations to appear in the first place, however, and subsequently not be obliterated by arbitrage, some category of investors must introduce some form of "demand shock", that can only result from some departure from von Neumann-Morgenstern utility. Here, we consider only rational behavior so that no opportunities for (costly) arbitrage arise in equilibrium.

[10] Empirical work has also been done by Chordia *et al.* (2008) and others to track the dynamics of liquidity as it moves from one category of assets to another. In the present paper, the menu of assets is too limited to throw any light on the evidence presented by these papers.

tion CAPM. These are the drags on expected-return that empiricists would encounter as a result of the presence of transactions costs.

1 Problem statement: the objective of each investor and the definition of equilibrium

We start with a population of two investors $l = 1, 2$ and a set of exogenous time sequences of individual endowments $\{e_{l,t} \in \mathbb{R}_{++}; l = 1, 2; t = 0, ...T\}$ on a tree or lattice. For simplicity, we consider a binomial tree so that a given node at time t is followed by two nodes at time $t + 1$ at which the endowments are denoted $\{e_{l,t+1,u}, e_{l,t+1,d}\}$. The transition probabilities are denoted $\pi_{t,t+1,j}$ $(\sum_{j=u,d} \pi_{t,t+1,j} = 1)$.[11] Notice that the tree accommodates the exogenous state variables only.[12] [13]

In the financial market, there are two securities, defined by their payoffs $\{\delta_{t,i}; i = 1, 2; t = 0, ..T\}$.[14] The "ticker" prices of the securities, which are not always transactions prices, are denoted: $\{S_{t,i}; i = 1, 2; t = 0, ...T\}$. The ticker price is an effective transaction price if and when a transaction takes place but it is posted all the time by the Walrasian auctioneering computer (which works at no cost).

Financial-market transactions entail deadweight, physical transactions costs. No one gets to pocket them. When an investor sells one unit of security i, turning it into consumption good, he receives the price reduced by $\varepsilon_{i,t}$ units of consumption goods and the buyer of the securities must give up $\lambda_{i,t}$ units. With symbol $\theta_{l,t,i}$ standing for the number of units of Security i in the hands of Investor l *after* all transactions of time t, Investor l solves the following problem:[15]

$$\sup_{\{c_l, \theta_l\}} \mathbb{E}_0 \sum_{t=0}^{T} u_l\left(\widetilde{c}_{l,t}, t\right)$$

subject to:

- terminal conditions:

$$\theta_{l,T,i} = 0,$$

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[11] Transition probabilities generally depend on the current state but we suppress that subscript.

[12] As has been noted by Dumas and Lyasoff (2010), because the tree only involves the exogenous endowments, it can be chosen to be *recombining* when the endowments are Markovian, which is a great practical advantage compared to the global-solution approach, which would require a tree in which nodes must be distinguished not just on the basis of the values of the exogenous variables but also the endogenous ones.

[13] It would be straightforward to write the equations below for more agents and more complex trees. The implementation of the solution technique is much more computationally intensive with more than two agents while it is not more complicated with a richer tree.

[14] It so happens that, without transactions costs, the market would be complete. But the derivations and the solution technique depend neither on the number of branches in the tree, nor on the number of securities. We could solve for the equilibrium with transactions costs in a market that would be incomplete to start with.

[15] The tilda ˜ is a notation we use to refer to a random variable.

- a sequence of flow budget constraints:

$$\begin{aligned} c_{l,t,j} + \sum_{i=1,2} \left[\theta_{l,t,i} - \theta_{l,t-1,i}\right]^{+} \left(S_{t,i} + \lambda_{i,t}\right) + \sum_{i=1,2} \left[\theta_{l,t,i} - \theta_{l,t-1,i}\right]^{-} \left(S_{t,i} - \varepsilon\right) & \\ = e_{l,t} + \sum_{i=1,2} \theta_{l,t-1,i}\delta_{t,i}; \forall t & \qquad (1) \end{aligned}$$

- and given initial holdings:[16]

$$\theta_{l,-1,i} = \bar{\theta}_{l,,i} \qquad (2)$$

In the flow budget constraint, the term $\sum_{i=1,2} \left[\theta_{l,t,i} - \theta_{l,t-1,i}\right]^{+} \left(S_{t,i} + \lambda_{i,t}\right)$ reflects the cost of purchases and the term $\sum_{i=1,2} \left[\theta_{l,t,i} - \theta_{l,t-1,i}\right]^{-} \left(S_{t,i} - \varepsilon\right)$ captures the proceeds of sales of securities.

The dynamic programming formulation of the investor's problem is:[17]

$$J_l \left(\left\{\theta_{l,t-1,i}\right\}, \cdot, e_{l,t}, t\right) = \sup_{c_{l,t}, \left\{\theta_{l,t,i}\right\}} u_l \left(c_{l,t}, t\right) + \mathbb{E}_t J_l \left(\left\{\theta_{l,t,i}\right\}, \cdot, \widetilde{e}_{l,t+1}, t+1\right)$$

subject to the flow budget constraint written at time t only.

Writing $\theta_{l,t,i} = \widehat{\theta}_{l,t,i} + \widehat{\widehat{\theta}}_{l,t,i} - \theta_{l,t-1,i}$, one can reformulate the same problem to make it more suitable for mathematical programming:

$$\begin{aligned} J_l \left(\left\{\theta_{l,t-1,i}\right\}, \cdot, t\right) = & \sup_{c_{l,t}, \left\{\widehat{\theta}_{l,t,i}, \widehat{\widehat{\theta}}_{l,t,i}\right\}} u_l \left(c_{l,t}, t\right) \qquad (3) \\ & + \mathbb{E}_t J_l \left(\left\{\widehat{\theta}_{l,t,i} + \widehat{\widehat{\theta}}_{l,t,i} - \theta_{l,t-1,i}\right\}, \widetilde{e}_{l,t+1}, t+1\right) \end{aligned}$$

subject to:

$$\begin{aligned} & c_{l,t} + \sum_{i=1,2} \left(\widehat{\theta}_{l,t,i} - \theta_{l,t-1,i}\right) \left(S_{t,i} + \lambda_{i,t}\right) \\ & + \sum_{i=1,2} \left(\widehat{\widehat{\theta}}_{l,t,i} - \theta_{l,t-1,i}\right) \left(S_{t,i} - \varepsilon_{i,t}\right) \qquad (4) \\ = \; & e_{l,t} + \sum_{i=1,2} \theta_{l,t-1,i}\delta_{t,i} \\ \widehat{\widehat{\theta}}_{l,t,i} \; \leq \; & \theta_{l,t-1,i} \leq \widehat{\theta}_{l,t,i} \qquad (5) \end{aligned}$$

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[16] It is assumed that $\sum_{l=1,2} \bar{\theta}_{l,,i} = 0$ or 1 depending on whether the security is assumed to be in zero or positive net supply.

[17] The form $J_l \left(\left\{\theta_{l,t-1,i}\right\}, \cdot, e_{l,t}, t\right)$ in which the value function is written refers explicitly only to investor l's individual state variables. The complete set of state variables actually used in the backward induction is chosen below.

Definition 1 *An equilibrium is defined as a process for the allocation of consumption* $c_{l,t}$*, a process for securities prices* $\{S_{t,i}\}$ *such that the supremum of (3) is reached for all* l, i, j *and* t *and the market-clearing conditions:*[18]

$$\sum_{l=1,2} \theta_{l,t,i} = 0 \text{ or } 1; t = 1, ...T; i = 1, ..2 \tag{6}$$

are also satisfied with probability 1 at all times $t = 1, ...T$.

In Appendix A, we show, using a shift of equations proposed in the context of incomplete markets by Dumas and Lyasoff (2010), that the equilibrium can be calculated, for given initial values of some endogenous state variables, which are the dual variables $\{\phi_{l,t}\}$, $\{R_{l,t,i}\}$ – as opposed to given values of the original state variables, viz., initial positions $\{\theta_{l,t-1,i}\}$ –, by solving the following equation system written for $l = 1, 2; j = u, d; i = 1, 2$. The shift of equations amounts from the computational standpoint to letting investors at time t plan their time-$t+1$ consumption $c_{l,t+1,j}$ but choose their time-t portfolio $\theta_{l,t,i}$ (which will finance the time-$t+1$ consumption).[19]

1. First-order conditions for time $t + 1$ consumption:

$$u'_l\left(c_{l,t+1,j}, t+1\right) = \phi_{l,t+1,j}$$

2. The set of time-$t+1$ flow budget constraints for all investors and all states of nature of that time:

$$e_{l,t+1,j} + \sum_{i=1,2} \theta_{l,t,i}\delta_{t+1,i,j} - c_{l,t+1,j} - \sum_{i=1,2} \left(\theta_{l,t+1,i,j} - \theta_{l,t,i}\right)\left(R_{l,t+1,i,j} + S_{t+1,i,j}\right) = 0$$

3. The third subset of equations says that, when they trade them, all investors must agree on the prices of traded securities and, more generally, they must agree on the posted "ticker prices" inclusive of the shadow prices R that make units of paper securities less valuable than units of consumption. Because these equations, which, for given values of $R_{l,t+1,i,j}$, are linear in the unknown state prices $\phi_{l,t+1,j}$, restrict these to lie in a subspace, we call them the "kernel conditions":

$$\begin{aligned} &-R_{1,t,i} + \frac{1}{\phi_{1,t}} \sum_{j=u,d} \pi_{t,t+1,j} \times \phi_{1,t+1,j} \times \left(\delta_{t+1,i,j} + R_{1,t+1,i,j} + S_{t+1,i,j}\right) \\ &= -R_{2,t,i} + \frac{1}{\phi_{2,t}} \sum_{j=u,d} \pi_{t,t+1,j} \times \phi_{2,t+1,j} \times \left(\delta_{t+1,i,j} + R_{2,t+1,i,j} + S_{t+1,i,j}\right) \end{aligned} \tag{7}$$

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[18] One equates $\sum_{l=1,2} \theta_{l,t}$ to 0 or 1 depending on whether the security is or is not in zero net supply.

[19] u'_l denotes "marginal utility" or the derivative of utility with respect to consumption.

4. Definitions:
$$\theta_{l,t+1,i,j} = \widehat{\theta}_{l,t+1,i,j} + \widehat{\widehat{\theta}}_{l,t+1,i,j} - \theta_{l,t,i}$$

5. Complementary-slackness conditions:
$$\begin{aligned}(-R_{l,t+1,i,j} + \lambda_{i,t+1,j}) \times \left(\widehat{\theta}_{l,t+1,i,j} - \theta_{l,t,i}\right) &= 0 \\ (R_{l,t+1,i,j} + \varepsilon_{i,t+1,j}) \times \left(\theta_{l,t,i} - \widehat{\widehat{\theta}}_{l,t+1,i,j}\right) &= 0\end{aligned}$$

6. Market-clearing restrictions:
$$\sum_{l=1,2} \theta_{l,t,i} = 0 \text{ or } 1$$

7. Inequalities:
$$\widehat{\widehat{\theta}}_{l,t+1,i,j} \leq \theta_{l,t,i} \leq \widehat{\theta}_{l,t+1,i,j}; -\varepsilon_{i,t+1,j} \leq R_{l,t+1,i,j} \leq \lambda_{i,t+1,j};$$

This is a system of 24 equations (not counting the inequalities) where the unknowns are $\left\{c_{l,t+1,j}, \phi_{l,t+1,j}, R_{l,t+1,i,j}, \theta_{l,t,i}, \widehat{\theta}_{l,t+1,i,j}, \widehat{\widehat{\theta}}_{l,t+1,i,j}; l = 1,2; j = u,d\right\}$. This is a total of 24 unknowns. We solve the system by means of the Interior-Point algorithm, in the implementation of Armand *et al.* (2008).

Besides the exogenous endowments $e_{l,t+1,j}$, the "givens" are the time-t investor-specific shadow prices of consumption $\{\phi_{l,t}; l = 1,2\}$ and of paper securities $\{R_{l,t,i}; l = 1,2; i = 1,2\}$, which must henceforth be treated as state variables and which we refer to as "endogenous state variables". Actually, given the nature of the equations, the latter variables can be reduced to two state variables: $\phi_{2,t} \times (R_{2,t,i} - R_{1,t,i})$ and $\frac{\phi_{1,t}}{\phi_{1,t}+\phi_{2,t}}$.[20]

In addition, the securities' prices $S_{t+1,i,j}$ are obtained by backward induction (see, in Appendix A, the third equation in System (15)):

$$\begin{aligned}S_{t,i} &= -R_{l,t,i} + \frac{1}{\phi_{l,t}} \sum_{j=u,d} \pi_{t,t+1,j} \phi_{l,t+1,j} \times (\delta_{t+1,i,j} + R_{l,t+1,i,j} + S_{t+1,i,j}); \\ S_{T,i} &= 0; R_{l,T,i} = 0\end{aligned} \tag{8}$$

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[20] However, the former is not bounded so that a proper range cannot be defined. For that reason, we decided not to reduce the dimension of the state space maximally and to use three state variables: $R_{2,t,i} - R_{1,t,i}$, $\phi_{1,t}$, $\phi_{2,t}$, the first of which is naturally bounded. The two variables $\phi_{1,t}$ and $\phi_{2,t}$ are one-to-one related to the consumption shares of the two investors, so that consumption scales are actually used as state variables. Consumption shares of the two agents do not add up to 1 because of the deadweight loss in transactions costs.

and the future positions $\theta_{l,t+1,i,j}$ (satisfying $\sum_{l=1,2}\theta_{l,t+1,i,j}=0$ or $1; i=1,..2$) are also obtained by an obvious backward induction of $\theta_{l,t,i}$, the previous solution of the above system, with terminal conditions $\theta_{l,T,i}=0$.[21]

Moving back through time till $t=0$, the last portfolio holdings we calculate are $\theta_{l,0,i}$. These are the portfolios held by the investors as they *exit* time 0. We need to translate these into *entering* portfolios holdings so that we can meet the initial conditions (2). The way to do that is explained in Appendix B.

2 Equilibrium asset holdings and prices at the initial point in time

In our benchmark setup, we consider two investors who have isoelastic utility and have different coefficients of relative risk aversion. One of them only (Investor $l=1$) receives a flow endowment. The desire to trade arises from the difference in the endowments and the differences in risk aversion.

As for securities, the subscript $i=1$ refers to a short-lived riskless security in zero net supply and the subscript $i=2$ refers to equity *also in zero net supply.* We call "equity" a long-lived claim that pays the endowment of Investor 1 ($\delta=e_1$). Transactions costs are levied on trades of equity shares (the "less liquid" asset); none are levied on trades of the riskless asset, which is also, therefore, the "more liquid" asset. The economy is of a finite-horizon type with $T=50$. The single exogenous process is the endowment process of the first investor, which is represented by a binomial tree mimicking a geometric Brownian motion.

The numerical illustration below cannot in any way be seen as being calibrated to a real-world economy.[22] Indeed, as has been noted in the introduction, investors in our model trade because they have differing risk aversions while they have access only to a menu of linear assets. The imbalanced portfolios they have to hold because of transactions costs act as an inventory cost similar to the cost

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[21] Equation (8) is based on one additional but inocuous assumption. At the terminal date, after all payments have been made and all goods consumed, securities have zero price but are still nominally held by someone; we assume that no transactions costs are levied on the virtual liquidation trades that take place at zero price at date T.

[22] In this pure-exchange general-equilibrium economy, where equity is in zero net supply, total consumption is equal to total endowments. And, in order to limit the number of exogenous processes, we have set dividends on the zero-net supply equity equal to the endowment. In order to capture some properties of real-world equity, we choose a process for all of these that reflects the behavior of dividends. The following set of papers document dividend dynamics. Lettau and Ludvigson (2005) write: "An inspection of the dividend data from the CRSP value-weighted index [] reveals that [] the average annual growth rate of dividends has not declined precipitously over the period since 1978, or over the full sample. The average annual growth rate of real, per capita dividends is in fact higher, 5.6%, from 1978 through 1999, than the growth rate for the period 1948 to 1978. The annual growth rate for the whole sample (1948-2001) is 4.2%." Volatility is reported to be 12.24%. Earlier evidence includes Campbell and Shiller (1988) who report for periods up to 1986 dividend growth rates of around 4%. Recently, van Binsbergen and Koijen (2010) estimate a growth rate of 5.89%.

A good mean value given this evidence is then probably to use a drift of 4.5% with a volatility of 13%.

incurred in inventory-management model of the Ho-and-Stoll (1980, 1983) variety. But our model does not include two other motives for trading that are obviously present in the real world such as the liquidity-trading motive (arising missing securities and endowment shocks that would be incompletely hedgeable even in a market without transactions costs) and the speculative motive (arising from informed trading due either to private signals or to differences of opinion). Above all, we have two traders, not millions. For these reasons, although our goal is to capture a higher-frequency motive to trade, the amount of trading we are able to generate is not sufficient to match high-frequency data quantitatively. We, therefore, keep a yearly trading interval because we need to cover a sufficient number of years to get some reasonable amount of trading. Even so, we are going to document interesting patterns that match real-world data qualitatively.

In order to save on the total amount of computation, we assume that the rate at which transactions costs per share traded are levied is proportional to the economy's endowment. Table 1 displays the per share transactions costs as a percentage of the endowment, whose initial value is equal to 1. This allows us to develop a property of scale invariance: all the nodes of a given point in time, which differ only by their value of the exogenous variable, are isomorphic to each other, where the isomorphy simply means that we can factor out the endowment. In this way, we do not need to perform a new calculation for each node of a given point in time; one suffices.[23]

Table 1 shows all the parameter values.[24] The risk aversion of Investor 1 is lower than that of Investor 2, so that Investor 1 is a natural borrower, as far as the riskless short-term security is concerned.

The endowment value at the initial point in time is set at 1 consumption unit and the equilibrium price of the stock that will pay a dividend for 50 years is found to be near 25.6 consumption units (see Figure 4, Panel (b)). If expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock. Therefore, our benchmark transactions cost of 5%×endowment×number of shares traded amounts to approximately equal to $5\%/25.6 \simeq 0.2\%$ of the value of the shares traded. *In what follows, the reader should interpret in this light every graph with a scale marked "Transactions Costs".*

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[23] This property, which we prove in Appendix C, holds even though investors have different risk aversions. Remarkably, the property is valid when $R_{2,t,i} - R_{1,t,i}$, $\phi_{1,t}$, $\phi_{2,t}$ are used as endogenous state variables of the backward recursion. With different risk aversions across investors, *it would not have held* if the endogenous state variables had been $\left\{\theta_{l,t-1,i}\right\}$, the portfolios held when entering each point in time t.

[24] The initial holdings of equity $\bar{\theta}_{1,2}$ by Investor 1 are just that. Separately, Investor 1 receives his/her endowment, which is the same stream of consumption units as the equity stream. In total, when $\bar{\theta}_{1,2} = -0.3$, Investor 1 collects 70% of the endowment stream.

Table 1: **Parameter Values and Benchmark Values of the State Variables.** This table lists the parameter values used for all the figures in the paper. The table also indicates the benchmark values of state variables, which are reference values taken by all state variables except for the particular one being varied in a given graph.

Name	**Symbol**	**Value**	**Range**
Parameters for exogenous endowment			
Horizon of the economy	T	50 years	
Expected growth rate of endowment		3.9%/year	
Time step of the tree		1 year	
Volatility of endowment		16.2%/year	
Initial endow. at $t = 0$ (cons. units)		1	
Parameters for the investors			
Investor 1's risk aversion	γ_1	2	
Investor's risk aversion	γ_2	4	
Investor 1's time preference	β_1	0.975	$[0.95, 0.99]$
Investor 2's time preference	β_2	0.975	
Transactions costs (as a fraction of endowment) per equity share traded			
When buying and when selling	$\lambda/e_1 = \varepsilon/e_1$	5%	$[0.01\%, 10\%]$
Benchmark values of the variables			
Initial hold. of riskless asset by Inv. 1	$\bar{\theta}_{1,1}$	-5	
Initial holding of equity by Investor 1.	$\bar{\theta}_{1,2}$	-0.3	

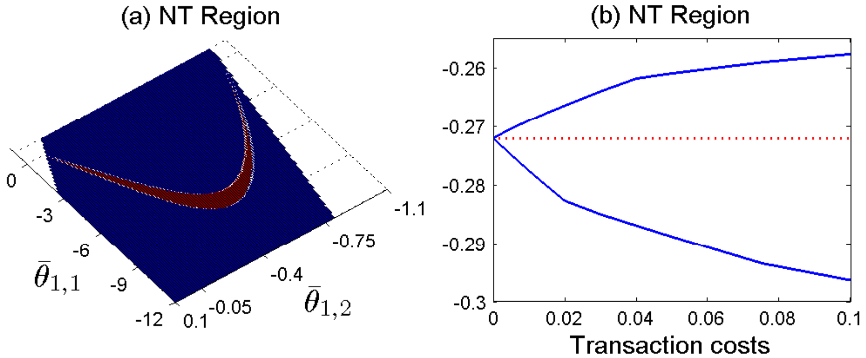
2.1 Equilibrium asset holdings

It is well-known from the literature on non-equilibrium portfolio choice that proportional transactions costs cause the investors to tolerate a deviation from their preferred holdings. The zone of tolerated deviation is called the "no-trade region". In previous work, the no-trade region had been derived for a given stochastic process of securities prices. We now obtain the no-trade region in general equilibrium, when two investors make analogous portfolio decisions and prices are set to clear the market.

2.1.1 Equilibrium no-trade region

Figure 1, Panel (a) plots the no-trade region for the different values of the initial holdings of securities. The lighter grey zone is specifically the no-trade region while the darker zone is the trade region. When, the holdings with which Investor 1 enters the trading date are in the trade region, the investors trade to reach the edge of the no-trade region; to the contrary, when the holdings upon entering the trading date are within the no-trade region, the investors do nothing. The crescent shape of the no-trade zone is the result of the difference in risk aversions between the two investors: there exists a curve (not shown) inside the zone which would be the locus of holdings in a frictionless, complete market. The white zone of the figure, on both sides of the dark grey zone, is not admissible; when entering holdings are in that zone, there exists no equilibrium as one investor would, at equilibrium prices, be unable to repay his/her negative positions to the other investor.Panel (b) of the same figure displays, for the benchmark values of the variables, the width of the no-trade region against the rate of transactions costs. Panel (c) illustrates how the shadow prices vary across the trade and no-trade regions: in one trade region, the shadow price per unit of endowment of one investor is equal to λ (the buy transaction cost per unit of endowment) while the other investor's shadow is equal to $-\varepsilon$ (the sell transaction cost) and in the other trade region, the opposite is true. The difference between their two shadow prices is, therefore, $\lambda+\varepsilon = 10\%$ or -10%. Within the no-trade region the difference is between these two numbers, with a discrete-version of the smooth-pasting condition holding on the optimal boundary and causing the shadow-price difference to taper off smoothly. The result is analogous to the no-trade region and the relative price of the equilibrium shipping model of Dumas (1992), with the difference that the trades considered are not costly arbitrages between *geographic locations* in which physical resources have different prices but are, instead, costly arbitrages between *people* whose private valuations of paper securities differ.

Figure 2 shows, against the rate of transactions costs, the holdings of the stock and bond with which Investor 1 exits a trading period in which he enters with initial holdings $(-5, -0.3)$. While this investor, who is less risk averse, is a natural borrower and thus chooses negative positions in the bond, increased transactions costs induce him to carry on with a smaller holding of equity. For that reason, he has to borrow less.



(c) Shadow Price Difference: $R_{1,1} - R_{2,1}$

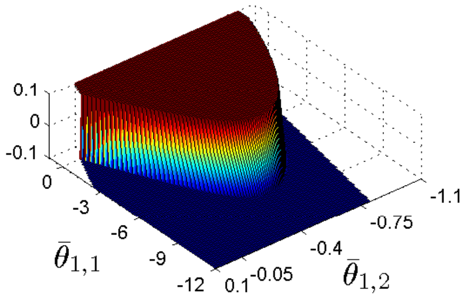


Figure 1: **Equilibrium no-trade region.** Panel (a) shows the no-trade region for different "entering" positions $\bar{\theta}$ of the agents. Transactions costs are equal to 5%× endowment × #shares, while Panel (c) displays the shadow prices across the trade and no-trade regions. Panel (b) shows the no-trade region for different levels of transaction costs from 0% to 10%. Consumption shares are set at the value corresponding to the initial holdings of Table 1. In all panels, parameters are as in Table 1. Recall that, if expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock.

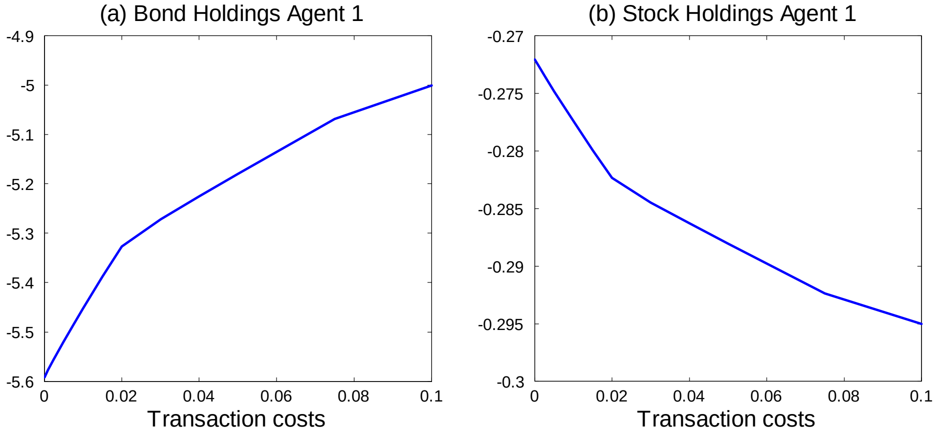


Figure 2: **Optimal "exiting" holdings θ of the securities**. Optimal bond and stock holdings of the first agent for different levels of transactions costs, in the range from 0% to 10%. All parameters and variables are set at their benchmark values indicated in Table 1 (entering holdings $(-5, -0.3)$). Recall that, if expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock.

2.1.2 The clientele effect

Do more patient investors hold less liquid assets as in the "clientele effect" of Amihud and Mendelson (1986)? We now vary the patience parameter of the first investor between 0.95 and 0.99. Figure 3 provides a clear illustration of the clientele effect: as Investor 1 becomes more patient, he/she holds more of the stock, which is the illiquid security and less of the short-term bond, which is the more liquid one. The result, however, depends very much on the initial holdings, here assumed to be 0 of the short-term bond and -0.5 of the stock.

2.2 Asset prices

According to Amihud and Mendelson (1986a, Page 228), the price of a security in the presence of transactions costs is equal to the present value of the dividends to be paid on that security minus the present value of transactions costs subsequently to be paid by someone currently holding that security. A similar conclusion was reached by Vayanos (1998, Page 18, Equation (31)) and Vayanos and Vila (1999, Page 519, Equation (5.12)).

There are many differences between our setting and the setting of Amihud and Mendelson. They consider a large collection of risk-neutral investors each of whom faces different transactions costs and are forced to trade. We consider two investors who are risk averse, face identical trading conditions and trade opti-

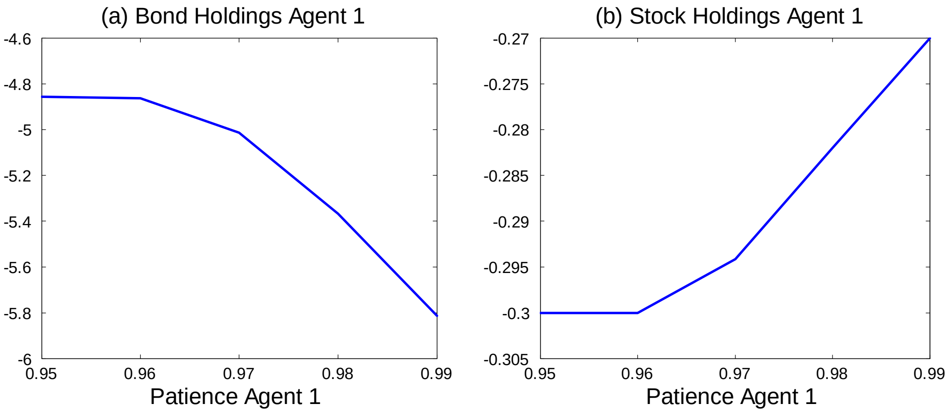


Figure 3: **Clientele effect.** Optimal bond and stock holdings of the first agent for different levels of patience, in the range from 0.95 to 0.99. All other parameters are as described in Table 1. Especially, transactions costs equal 5% per equity share traded.

mally. Nonetheless, their statement is an appealing conjecture to be investigated using our model.

Recall from Equation (8) that the securities' ticker prices $S_{t+1,i,j}$ are:

$$\begin{aligned} S_{t,i} &= -R_{l,t,i} + \mathbb{E}_t\left[\frac{\phi_{l,t+1}}{\phi_{l,t}} \times (\delta_{t+1,i} + R_{l,t+1,i} + S_{t+1,i})\right]; \\ S_{T,i} &= 0; R_{l,T,i} = 0 \end{aligned}$$

where the terms $R_{l,t,i}$ $(-\varepsilon_{i,t} \leq R_{l,t,i} \leq \lambda_{i,t})$ capture the effect of current and anticipated trading costs.

We now present two comparisons. First, we compare equilibrium prices to the present value of dividends on security i calculated at the equilibrium state prices *under transactions costs* of Investor l. We denote this private valuation $\hat{S}_{t,i,l}$:

Definition 2

$$\hat{S}_{t,i,l} \triangleq \frac{1}{\phi_{l,t}} \sum_{j=u,d} \pi_{t,t+1,j}\phi_{l,t+1,j} \times \left(\delta_{t+1,i,j} + \hat{S}_{t+1,i,j}\right); \hat{S}_{T,i} = 0$$

We show that:

Proposition 3

$$S_{t,i} = \hat{S}_{t,i,l} - R_{l,t,i} \tag{9}$$

which means that the ticker prices of securities can at most differ from the present value of their dividends as seen by Investor l by the value of the transactions costs incurred or imputed by Investor l at the current trading date only. Figure 4, Panel (b) plots the ticker price and the present value of dividends for different values of transactions costs, thus illustrating the decomposition of Equation (9). The result is as expected from our analytical derivations. The difference between them is in the range $[-5\%, +5\%]$ of endowment, where we achieve the boundaries of this range when the system hits the boundaries of the trade region. For the no-trade region, it is somewhere within the range.

Proof. In Appendix D ■

Second, we compare equilibrium asset prices that prevail in the presence of transactions costs to those that would prevail in a frictionless economy, based, that is, on state prices that would obtain *under zero transactions costs.* Denoting all quantities in the zero-transactions costs economy with an asterisk *, and defining:

$$\Delta\phi_{l,t} \triangleq \frac{\phi_{l,t}}{\phi_{l,t-1}} - \frac{\phi^*_{l,t}}{\phi^*_{l,t-1}}$$

we show that:

Proposition 4

$$S_{t,i} = S^*_{t,i} - R_{l,t,i} + \mathbb{E}_t\left[\sum_{j=t+1}^{T} \frac{\phi_{l,j-1}}{\phi_{l,t}} \Delta\phi_{l,j}\left(\delta_j + S^*_j\right)\right] \tag{10}$$

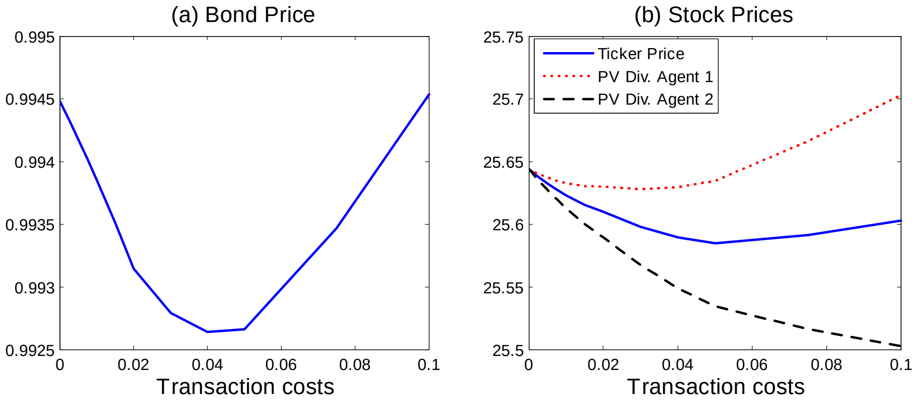
Proof. In Appendix E ■

That is, the two asset prices differ by two components: (i) the current shadow price $R_{l,t}$ of which we know that it is at most as big as the one-way transactions costs, (ii) the present value of all future price differences arising from the change in state prices and consumption induced by the presence of transactions costs.

While the ticker price S and the present value of dividends $\hat{S}$ differ from each other at most by one round of transactions costs, both of them are reduced by the presence of transactions costs because, over some range, the state prices ϕ are lower with transactions costs than without them. Panel (c) of the same figure illustrates the decomposition of Equation (10).

The reason for the drop is not that the investors pay big amounts of transaction costs in the future but that they do not hold the optimal frictionless holdings and, therefore, also have consumption schemes that differ from those that would be optimal in the absence of transactions costs. The differences in consumption schemes then influence the future state prices and accordingly the present values of dividends. Beyond some level of transactions costs, however, the effect of state prices on the initial stock price reverses itself because a general reduction of the future consumption level increases marginal utilities while, as a matter of definition, equity payoffs are left untouched. Vayanos (1998) had already noted that prices can be higher in the presence of transactions costs.

Because the affected state prices are applied by investors to all securities, the change in the state prices is also reflected in the one-period bond price which varies (non monotonically) as we vary the transactions costs applied to equity, as is illustrated in Panel (a) of Figure 4.



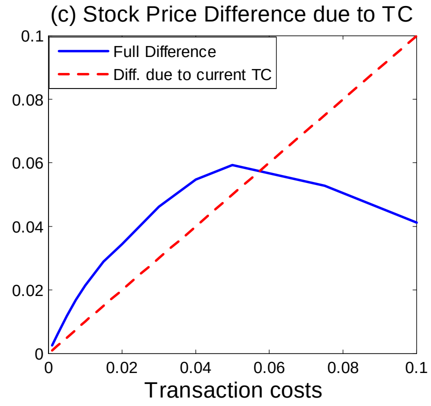


Figure 4: **Initial asset prices.** Panel (a) shows the initial period's bond price for different levels of transactions costs in the range from 0% to 10%. All parameters and variables are set at their benchmark values indicated in Table 1 (entering holdings $(-5, -0.3)$). Panel (b) shows the initial period's stock price and the two agents' present values of dividends $\hat{S}_{t,i,l}$ for different levels of transactions costs. Panel (c) shows the difference between the initial stock price in an economy with transactions costs and the stock price in economies without transactions costs. In addition, we show the component of the stock price difference that is due to the current amount of transactions costs. Recall that, if expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock.

3 Time paths of prices and holdings

We now study the behavior of the equilibrium over time and the transactions that take place. Figure 5 displays a simulated sample path illustrating how our financial market with transactions costs operates over time. In an attempt to remove the effects of the finite horizon on trade decisions, we only display the first 25 periods, although the economy runs for 50 periods ($T = 50$).[25]

Panel (a) shows a sample path of: (i) stock holdings as they would be in a zero-transaction cost economy, (ii) the actual stock-holdings with a 5% transaction cost[26] and (iii) the boundaries of the no-trade zone, which fluctuate over time. The boundaries fluctuate very much in parallel with the optimal frictionless holdings, allowing a tunnel of deviations on each side. Within that tunnel, the actual holdings move up or down whenever they are pushed up or down by the movement of the boundaries, with a view to reduce transactions costs and making sure that no unnecessary round trip ever occurs. These three graphs clarify the logic behind the actual holdings

Panel (b) shows the stock ticker price (expressed in units of the consumption good), with transaction dates highlighted by a circle. While the ticker price forms a stochastic process with realizations at each point in time, transactions prices materialize as a "point process". Panel (c) displays the individual-investor valuations (i.e., present values of dividends) as a difference with the ticker price, as in decomposition (9). The ticker price is thus seen as an average of the two private valuations. When the two valuations differ by more than the sum of the one-way transaction costs for the two investors, a transaction takes place. The direction of the trade depends, of course, on the sign of the difference between private valuations. The increments in the private valuations of Investor 1 are more highly correlated with the increments in the ticker price than those of Investor 2. In fact, Investor 2 does not buy on an up move in the ticker price. In our benchmark example, Investor 1 has a lower risk aversion. Although ours remains a Walrasian market and not a dealer market, Investor 1 is closer to the proverbial "market maker" of the Microstructure literature, who is traditionally assumed to be risk neutral, and Investor 2 may be viewed as a "customer". If we wanted to push the analogy further, we could define the "bid" and the "ask" prices as being equal to Investor 1's private valuation plus and minus transactions costs and we would call a purchase by Investor 2 a "buy".[27]

Panel (d) of the figure shows the fluctuations of Amihud's LIQ measure, which is defined below. It will be useful to us later on.

Finally, Panels (e) and (f) illustrate decomposition (10) over time. Deviations are here expressed relative to the price that would prevail if transactions costs

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[25] If the equilibrium of this economy had been a stationary one, it would also have been useful to introduce a number of "run-in" periods, in an attempt to render the statistical results of this section independent of initial conditions. But, with investors of different risk aversions, no equilibrium is stationary.

[26] Again, this is 5% of the number of shares, not the value of shares traded.

[27] The pattern is reminiscent of Lee and Ready (1991) but would be opposite to their rule. When, in empirical work, the direction of trade is not observed, they recommend to classify the transaction as a buy (by the customer) if it occurs on an "uptick".

were zero. For example, for the bond, the quantity is: $\frac{S^*_{1,t}-S_{1,t}}{S^*_{1,t}}$ where $S^*_{1,t}$ denotes the price in a zero-transactions cost economy. The price of the stock is almost always reduced by the presence of transactions costs but such is not the case for the bond. Panel (f) shows, again in relative terms, the components of the difference between the stock price in a zero transactions costs economy and an economy with transactions costs, along the same path, as seen by Investor 1. The components are due to the current amount of (shadow or actual) transactions costs and to the future difference in pricing (state prices).

The number of shares traded is such that the current amount of transactions costs hardly ever exceeds 25bp of the value of the stock.[28]

We now demonstrate some properties of the sample paths. We first investigates univariate properties of trades on the one hand and of asset price increments on the other. Then we investigate bivariate properties of trades and price changes.

3.1 Trades over time

We examine the trading volume and the waiting times between trades. The trading volume is defined as the sum of the absolute values of changes in θ_2 (shares of the stock) over the first 25 periods of the tree. The average trading volume is shown in Figure 6, Panel (a) and is as one would expect it to be; it decreases with transactions costs. Correspondingly, the average waiting (Panel (b)) between trades rise. We also show in Panel (c) the volatility of the waiting time, which is a first measure of the (endogenous) liquidity risk that the investor has to bear because he/she operates in a market with friction. We examine in section 4 below how this risk is priced.

The microstructure literature has established that trades are autocorrelated and the order flow is predictable (Hasbrouck (1991a, 1991b) and Foster *et al.* (1993). Looking at the time path in Panel (a) of Figure 5, we have already pointed out that the investors smooth their trades over time in order to keep transactions costs low. We investigate the matter more systematically in Figure 7 which displays the average of 20000 simulations. The microstructure literature usually ascribes the autocorrelation of trades to a trader's desire to avoid price impact by, for instance, breaking up large trades into smaller ones, a form of behavior known as "order fragmentation". But, here we see that the desire simply to avoid unnecessary round trips, in a purely Walrasian market, also leads to a strong autocorrelation.

3.2 Total and quadratic variation of prices

We are interested in determining in which way, as one decreases transactions costs, the point process of transactions prices approaches the process that would

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[28] Figure 6 below shows that, for a 5% transactions cost, which corresponds approximately to 20bp of the value of a trade, the average number of shares traded is approximately 0.22 per year (out of a total number of shares normalized to 1).

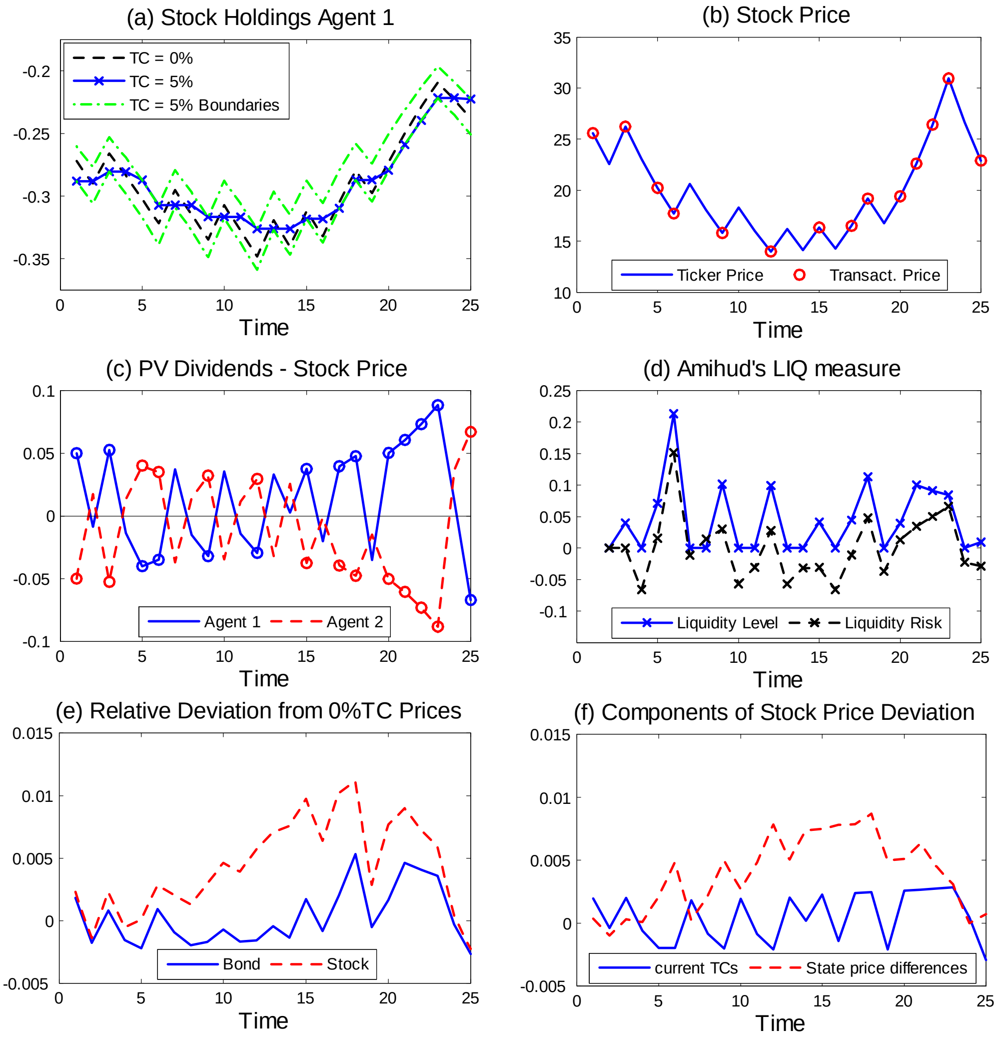

Figure 5: **Sample time paths of stock holdings, the stock price and the difference between the stock price and each investor's value of the present value of dividends**. Panel (a) shows stock holdings of the first agent along the paths for zero and 5% transactions costs per equity share traded. All parameters and variables are set at their benchmark values indicated in Table 1 (time-0 holdings $(-5, -0.3)$). Panel (b) shows the stock ticker price along the sample path for 5% transactions costs. Panel (c) shows the present values of future dividends for the two agents along the same path. Transactions are highlighted by a circle. Panel (d) shows Amihud's LIQ measure along the same path as well as its unanticipated or permanent component (marked Liquidity Risk). Panel (e) shows the relative deviation between the asset price in a zero transactions economy and an economy with transactions costs along the same path. Panel (f) shows the components of the difference between the stock price in a zero transactions costs economy and an economy with transactions costs, along the same path.

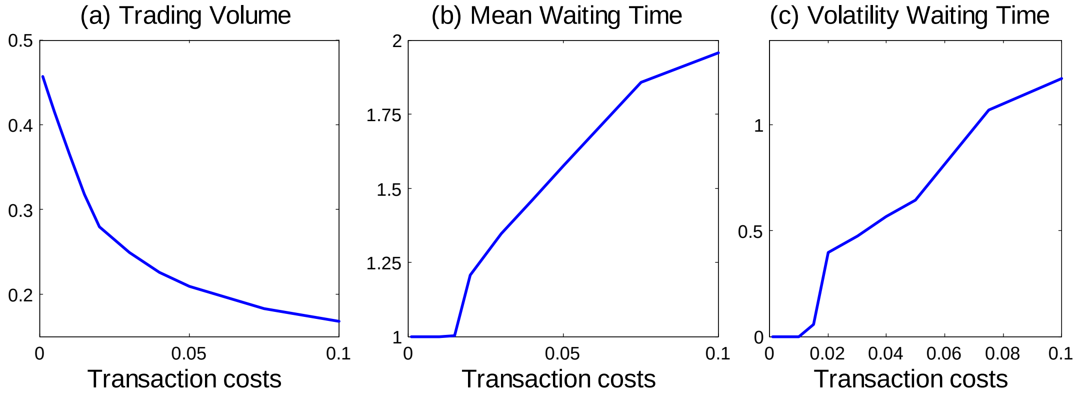

Figure 6: **Trading volume and waiting time against transactions costs**. Panels (a) and (b) show the average (across paths) stock trading volume/year and the average waiting time between trades (measured in years) respectively, up to period 25 for different levels of transactions costs, in the range from 0% to 10%. Panel (c) shows the standard deviaiton of the waiting time calculated the same way. All parameters and variables are set at their benchmark values indicated in Table 1. We use 20,000 simulations along the tree. Panel (c) shows the standard deviation of waiting time computed the same way. Recall that, if expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock.

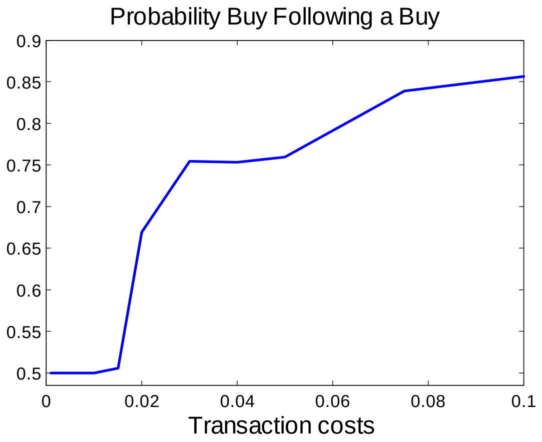

Figure 7: **Serial dependence of trades**: the frequency of a buy transaction following a previous buy transaction. All parameters and variables are set at their benchmark values indicate din table 1. We use 20,000 simulations along the tree. In each, the first 25 periods only are used. Recall that, if expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock.

prevail in the absence of transactions costs, which in the limit of continuous time would be a continuous-path process. As is well-known, the Brownian motion is characterized by the fact that its total variation, calculated over a finite period of time, is infinite while its quadratic variation is finite.[29] To discuss the limit, we generate many simulated paths of the stock price for zero transactions cost and calculate average (across paths) total variation and quadratic variation over the first 25 periods. Then we generate the same paths of transactions prices and holdings with transactions costs increasing to 10% and we calculate again average (across paths) total variation and quadratic variation. These are plotted against transactions costs in Figure 8.

The total variation of the ticker price is practically invariant to transactions costs. It is finite because this is a finite tree but, if one took the limit of continuous time, it would be infinite, as is the case for Brownian motions. When reducing transactions costs, transactions become more and more frequent and the total variation of the transactions prices rises rapidly to approach the total variation of the ticker price but then is capped by it. If one took the limit to continuous time, it would also approach infinity.

As can be expected from the nature of the tree, the quadratic variation of the ticker price is approximately constant (note the vertical scale). The quadratic

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[29] Total variation is the sum of the absolute values of the segments making up a path or connecting the dots, whereas quadratic variation is the sum of their squares.

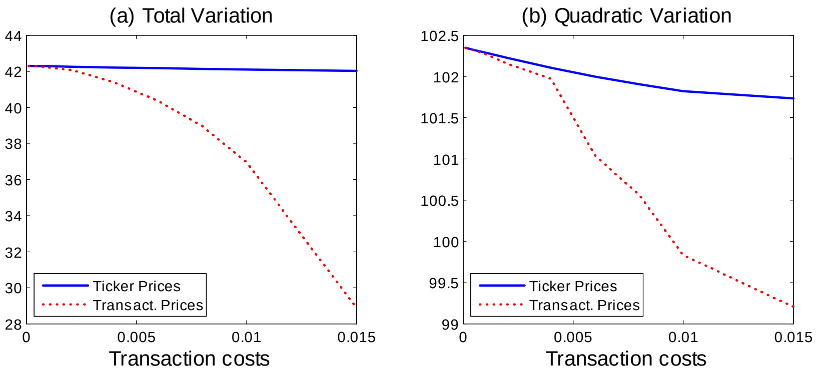

Figure 8: **Total and quadratic variations of stock price depending on transactions costs**. Panel (a) shows the total variation (defined in footnote 29) up to period 25 for different levels of transactions costs, in the range from 0% to 10%. All parameters and variables are set at their benchmark values indicate din table 1. We use 20,000 simulations along the tree. Panel (b) shows the quadratic variation computed the same way. Recall that, if expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock.

variation of the transactions prices rises modestly.

3.3 Joint behavior of transactions prices and trades

We now explore the joint behavior of prices and transactions, which is a favorite topic of the empirical Microstructure literature, aiming to measure the "price impact" of trades, when customer trades arrive randomly.[30] Much of the literature relates price impacts to traders' hedging and speculative motives (the latter arising from the presence of informed traders) and possibly also to their strategic behavior. We want to determine whether the empirical phenomena that have been unearthed could also be explained, in a more mundane fashion, by physical transactions costs and the heterogeneity of tastes of the investor population, when customers' orders do not arrive randomly but are, instead, those of intertemporally optimizing agents seeking to economize on the cost of transacting. Our findings are not meant to contradict the informed-trading interpretation offered by the Microstructure literature. Indeed, following Glosten and Milgrom (1985)'s dealership theory, we know that bid-ask spreads, which in the real world are a large component of transactions costs, are related to informed trading.

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[30] See the surveys by Biais et al. (2005), Amihud et al. (2005), the monographs by Hasbrouck (2007) and by de Jong and Rindi (2009), and the works of Roll (1984), Campbell et al. (1993), Llorente et al. (2002) and Sadka (2006).

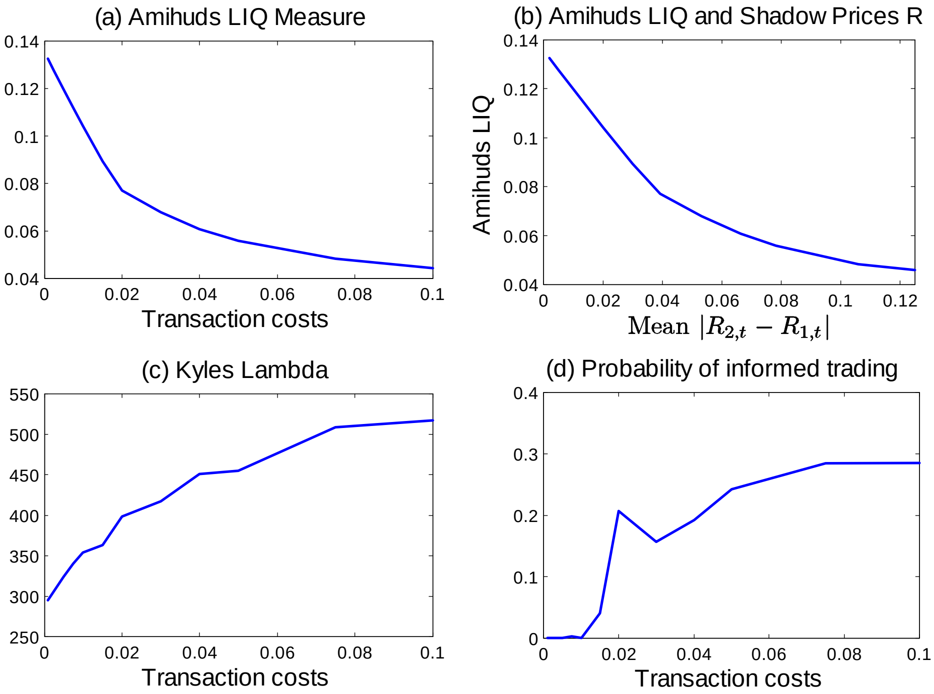

Figure 9: **Liquidity variables.** The first two panels show the average across paths of Amihud's LIQ measure, computed using simulated results up to period 25 for different levels of transactions costs, in the range from 0% to 10% (Panel (a)) and against the average shado wprice differential (Panel (b)). All parameters and variables are set at their benchmark values indicate din table 1. We use 20,000 simulations along the tree. Panel (c) shows Kyle's lambda and computed in the same way using Madhavan-Smidt regression. Panel (d) shows the average PIN measure. Recall that, if expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock.

We investigate the relationship between transactions costs and three popular measures of price impact. A popular measure is the ILLIQ measure of Amihud (2002). We interpret it as being equal to the average over time of the absolute values of the change in the ticker price divided by the contemporaneous absolute volume of trade. However, in most sample paths, there are node with zero trades. We prefer, therefore, to compute a LIQ measure equal to the average of volume of trade over the absolute price change. In Figure 5, Panel (d), we have exhibited the fluctuations of LIQ along a sample path, illustrating the way it varies with the volume of trade and the shadow prices of investors. But, as has been emphasized by Acharya and Pedersen (2005), Pástor and Stambaugh (2003) and Sadka (2006), the risk borne by an investor is not given by the totality of the fluctuations of LIQ but by the innovations in the process. For that reason, we have also shown in the same graph the sample path of the innovation in the LIQ process estimated as an AR(2) process. These are used below (in Section 4) in our discussion of liquidity pricing.

Figure 9, Panels (a) and (b) show how LIQ, which is commonly used to estimate effective trading costs, is, on average, related to the given one-way transactions costs of our model and to the average effective cost for the investors captured by their average shadow costs R difference (less than or equal to a two-way transaction cost). We compute, at each node where there is a trade, the price change since the last trade as well as the purchase or sale at that node and collect the ratios of those. We then compute the average. Panel (a) shows that this average LIQ is monotonically related to one-way trading costs. For Panel (b) we have, in addition, calculated an average across paths of the shadow cost differences); the panel again shows a monotonic relationship.

More formal methods to measure price impact are based on reduced forms of theoretical Microstructure models. Some are motivated by the desire to capture informed trading (Roll (1984), extended by Glosten and Harris (1988)). Others (Ho and Macris (1984)) are motivated by inventory considerations. Madhavan and Smidt (1991) run a regression which is meant to capture both effects. We implement their idea in the following way. At each node where there is a trade, we collect the price change since the last trade, the signed amount of purchase or sale by Investor 2 and the current equity holding of Investor 1. We then regress, across nodes of various times, the price change on these two variables. The responsiveness of price to order quantity, often referred to as Kyle's λ is displayed in Figure 9, Panel (c) against transactions costs. It is also mostly rising with transactions costs.

Finally, we calculate average PIN (Probability of Informed Trading). We implemented the procedure described in Easley *et al.* (2002). The parameters of the underlying sequential trade model are estimated by Maximum Likelihood. In empirical studies what is typically done is the following: given data for a specific horizon, say, a year, one counts the buy and sells on a pre-defined unit of time, e.g., day or week, so that one finally has a vector containing the number of buy and sell trades for each unit in the year. Here, we counted for each simulation the number of buys and sells and thus arrived at a vector of numbers of buy

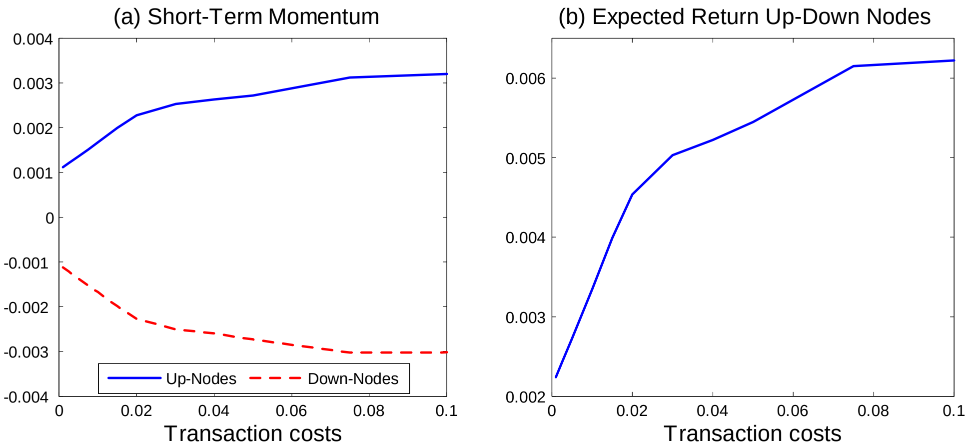

Figure 10: **Short-term momentum in ticker price returns:** we sort the nodes into two groups (dubbed "up" and "down" nodes) based on their last period return (realized return above or below the *ex ante* conditional return). The curves show the one-period ahead conditional expected return of the two groups of nodes. The computation is done on the first 25 periods of the tree. All parameters and variables are set at their benchmark values indicated in Table 1. Recall that, if expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock.

and sell trades for each simulation path.[31] Thus, we can simply estimate the parameters of the model for each level of transaction cost and compute the PIN measure. Panel (d) displays the results. For an economy without transactions costs the PIN measure is zero. If we increase transaction costs, the PIN measure quickly increases. The level of the PIN measure is also quite high.[32]

Overall, Figure 9 demonstrates that commonly used measures of price impact are not necessarily measures of the degree of informed trading present in the marketplace but could be the mechanical result of intertemporal optimization in the presence of market frictions.

3.4 "Anomalies": momentum and post-earnings announcement drift

We now investigate leading anomalies such as short-run momentum, the tendency of an asset's recent performance to continue into the near future, and the

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[31] This means we identify a simulation run with one unit in the empirical data and instead of having several successive units we have several simulation runs.

[32] For example, in Easley et al. (2002), the mean of the PIN measure is 0.2 with a typical 95% percentile of 0.3.

post-earnings announcement drift (PEAD).[33] We want to determine whether transactions costs might be an explanation for these empirically observed phenomena.

In order to elicit patterns over time, we sort the nodes into two groups (dubbed "up" and "down" nodes) based on their last period return (realized return above or below the *ex ante* conditional return). The curves show the one-period ahead risk-adjusted conditional expected return of the two groups of nodes. While risk adjustment in empirical work on this topic is typically performed using Sharpe's static CAPM, we have done it here using the classic consumption CAPM, which, absent liquidity corrections, would be the correct dynamic CAPM. Figure 10, Panel (a) shows that short-run momentum is present in our equilibrium and increasing with transactions costs.

The post-earnings announcement drift (PEAD) is the tendency of stocks' earning surprises to predict positively future risk-adjusted returns. Momentum and PEAD are one and the same phenomenon, with the only difference is that, for PEAD, one specifies what piece of news one is looking at that moves the price, whereas, for momentum, one considers any unspecified news item that moves the price. In the case of our model, we can regard up moves of the endowment as a positive earnings-announcement surprise but, in our model, that is the only possible piece of news anyway. So, there is really no difference since an up node always produces an up move in the price. Hence, Figure 10, Panel (b), representing the difference between the two curves of Panel (a) can be regarded as a match of the classic PEAD evidence presented, e.g., in Bernard and Thomas (1989). The risk-adjusted return differential can reach 60bp per year.

Thus we have demonstrated theoretically that momentum and PEAD phenomena appear in a financial market with frictions. This is consistent with Sadka (2006)'s empirical work, which showed that, if one corrects for liquidity premia, the momentum and PEAD puzzles disappear.

4 The pricing of liquidity and of liquidity risk

Based on a pure portfolio-choice reasoning, Constantinides (1986) argued that transactions costs make little difference to risk premia in the financial market. Liu and Lowenstein (2002) and Dumas and Puopolo (2010), still on the basis of portfolio choice alone, challenge that view by pointing out that the conclusion of Constantinides holds only when rates of return are identically, independently distributed (IID) over time. We go one step further than these authors, in that we now get the deviations in a full general-equilibrium model, when endowments are IID but returns themselves are not and investors must also face the uncertainty about the dates at which they can trade.

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[33] See, for example, Jegadeesh & Titman (1993) for evidence on short-run momentum.

4.1 Deviations from the classic CAPM under transactions costs

In our equilibrium, the capital-asset pricing model is Equation (8) above. The dual variables R (in addition to the intertemporal marginal rates of substitution ϕ) drive the prices of assets that are subject to transactions costs, as do, in the "LAPM" of Holmström and Tirole (2001), the shadow prices of the liquidity constraints.[34]

It can be rewritten as:[35]

$$\mathbb{E}_t\left[r_{l,t+1,i}\right] - r_{1,t+1} = -\text{cov}_t\left[r_{l,t+1,i}, \frac{\phi_{l,t+1}}{\mathbb{E}_t\left[\phi_{l,t+1}\right]}\right]; i \neq 1 \tag{11}$$

where:

$$1 + r_{l,t+1,i,j} \triangleq \frac{\delta_{t+1,i,j} + R_{l,t+1,i,j} + S_{t+1,i,j}}{S_{t,i} + R_{l,t,i}}$$

is the shadow return inclusive of transactions costs from the point of view of Investor l and where Asset #1 is the security that is riskless and not subject to transactions costs. It is to be noted that the *very definition of a rate of return* and, consequently, expected returns and risk premia differ across investors.

We go through a decomposition exercise similar to that performed by Acharya and Pedersen (2005). We break up the return into a component τ that is related to transactions costs and one $\hat{r}$ that is not so, or less so, related (although, literally speaking, both would be different in the absence of time $t+1$ transactions costs):

$$\begin{aligned}
\tau_{l,t+1,i,j} &\triangleq \frac{R_{l,t+1,i,j}}{S_{t,i} + R_{l,t,i}}; 1 + \hat{r}_{l,t+1,i,j} \triangleq \frac{\delta_{t+1,i,j} + S_{t+1,i,j}}{S_{t,i} + R_{l,t,i}} \\
1 + r_{l,t+1,i,j} &\equiv 1 + \hat{r}_{l,t+1,i,j} + \tau_{l,t+1,i,j}
\end{aligned}$$

Were it not for the presence of the shadow price $R_{l,t,i}$ in the denominator of its definition, a number which is small compared to the ticker price $S_{t,i}$, the rate of return $\hat{r}$ would be equal to the rate of return observable on the ticker tape. With that notation, the CAPM in our equilibrium is:[36]

$$\begin{aligned}
\mathbb{E}_t\left[\hat{r}_{l,t+1,i}\right] - r_{1,t+1} &= -\mathbb{E}_t\left[\tau_{l,t+1,i}\right] - \text{cov}_t\left[\hat{r}_{l,t+1,i}, \frac{\phi_{l,t+1}}{\mathbb{E}_t\left[\phi_{l,t+1}\right]}\right] \\
&\quad -\text{cov}_t\left[\tau_{l,t+1,i}, \frac{\phi_{l,t+1}}{\mathbb{E}_t\left[\phi_{l,t+1}\right]}\right]
\end{aligned}$$

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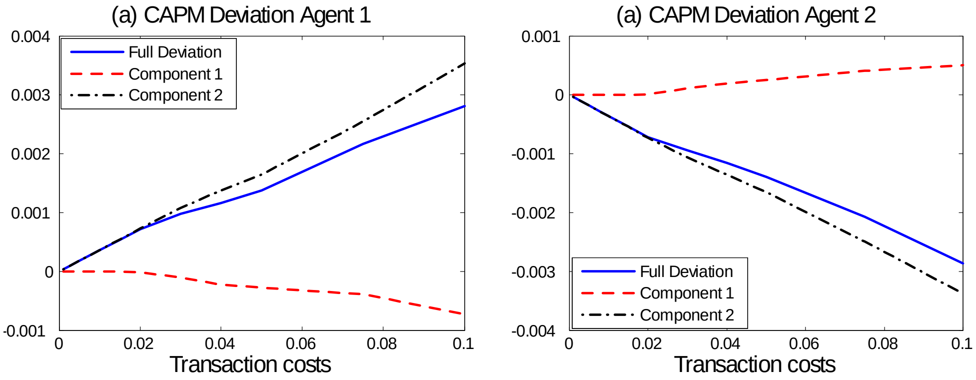
[34] Holmstrom and Tirole (2001) assume that their liquidity constraint is always binding. Here, we allow the inequality constraints (5) to hold whenever it is optimal for them to do so.

[35] Recall that the security numbered $i = 2$ is equity and the security numbered $i = 1$ is the short-term bond.

[36] In order to match even further the empirical findings of Acharya and Pedersen, we could further split $\frac{\phi_{l,t+1}}{\mathbb{E}_t\left[\phi_{l,t+1}\right]}$ into its value $\frac{\phi^*_{l,t+1}}{\mathbb{E}_t\left[\phi^*_{l,t+1}\right]}$ as it would be in the absence of transactions costs and a component related to transactions costs. That may not be necessary.


(a) CAPM Deviation Agent 1

(a) CAPM Deviation Agent 2

Figure 11: **CAPM deviations**. Panels (a) and (b) show the deviations from the classic consumption CAPM, computed using results up to period 25 for different levels of transactions costs, in the range from 0% to 10%. "Component 1" is the expected liquidity premium; "component 2" is the liquidity risk premium. All parameters and variables are set at their benchmark values indicated in table 1. Recall that, if expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock.

We now define deviations from the classic consumption CAPM that occur in our equilibrium as:

Definition 5

$$\begin{aligned} \textit{CAPM deviation} &\triangleq \mathbb{E}_t\left[\hat{r}_{l,t+1,i}\right] - r_{1,t+1} + cov_t\left[\hat{r}_{l,t+1,i}, \frac{\phi_{l,t+1}}{\mathbb{E}_t\left[\phi_{l,t+1}\right]}\right] \\ &= -\mathbb{E}_t\left[\tau_{l,t+1,i}\right] - cov_t\left[\tau_{l,t+1,i}, \frac{\phi_{l,t+1}}{\mathbb{E}_t\left[\phi_{l,t+1}\right]}\right] \qquad (12) \end{aligned}$$

Because it involves the shadow prices, the deviation is specific to each investor l. Equation (12) says that the deviation from the classic CAPM is the sum of expected transactions costs (or expected liquidity) and a premium for the liquidity risk created by transactions costs. Figure 11 Panels (a) and (b) show the deviations from the classic consumption CAPM, computed using simulated returns up to period 25 for different levels of transactions costs, in the range from 0% to 10%. "Component 1" is the expected liquidity premium $-\mathbb{E}_t\left[\tau_{l,t+1,i}\right]$; "component 2" is the liquidity risk premium $-\mathrm{cov}_t\left[\tau_{l,t+1,i}, \frac{\phi_{l,t+1}}{\mathbb{E}_t\left[\phi_{l,t+1}\right]}\right]$.

As expected, the absolute CAPM deviation is increasing in transactions costs. The CAPM deviation is positive for the first investor, i.e., the less risk-averse investor demands a higher expected return in an economy with transactions costs whereas the more risk-averse Investor 2 demands a lower expected return. This is due to the fact that the covariance between the first investor's pricing kernel and the τ return, i.e., liquidity risk, is positive.

In Amihud and Mendelson (1986a), it was explained that the total premium should be concave in the size of transactions costs. For that reason, Amihud and Mendelson (1986b) fitted the cross section of equity portfolio returns to the log of the bid-ask spread of the previous period and found a highly significant relationship. Our figure does not exhibit that concavity property.[37]

In our very simple benchmark setup, the deviation reaches 10 to 30bp, which is approximately equal to the transaction costs, measured as a percentage of the value of each trade. That deviation is too small to be able to account for the several percentage points of returns that are commonly attributed empirically to liquidity premia.

4.2 Liquidity and asset pricing

In our CAPM (11), rates of return are based on payoffs $\delta_{t+1,i,j} + R_{l,t+1,i,j} + S_{t+1,i,j}$ where the shadow prices that differentiate capital gains from consumption payoffs are present, as they are in the time t price $S_{t,i}+R_{l,t,i}$. These shadow prices are generally not observable. In empirical work, the rate of return on a security is commonly computed as $\frac{\delta_{t+1,i,j}+S_{t+1,i,j}}{S_{t,i}}$ between fixed, equally spaced calendar points in time, between which the security is held. However, the concept of holding period is quite arbitrary. Absent transactions costs, the only holding period that would make sense is one approaching zero, since investors are ready to trade at any time. Armed with the current model, we determine the holding period endogenously. In a model with more than two agents, holding periods would generally differ across people. With two agents, who can only trade with each other, the holding periods are identical across agents but, between trades, their desires to trade differ. That desire is reflected in the investor-specific shadow prices, which must be taken into account if rates of return continue to be based on fixed, equally spaced points in time. If one wanted to test our CAPM, one would not use the standard concept of rate of return measured between fixed points in time. Instead, one would use transactions prices only, which do not occur at fixed time intervals, and one would substitute out the values of prices that are unobserved for lack of transaction.[38] That, however, is not the way empirical tests have been conducted by previous authors.

In Acharya and Pedersen (2005) and Pástor and Stambaugh (2003), tests were conducted on a cross-section of monthly portfolio returns, looking at changes in market liquidity as a new risk factor. In Figure 5, we have displayed a sample path of the time variation of an empirical measure of liquidity and of its unanticipated component. They were seen to fluctuate widely over time. For that reason, liquidity fluctuations, in addition to current and expected liquidity,

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[37] See also Figure 3.1 in Amihud et al. (2005). The analogy between what we do and what they do is not perfect as they display a cross-section of firms affected differently by transactions costs and we display a single premium for different levels of transactions costs. But the underlying rationale is identical.

[38] See the discussion on page 89 of Hasbrouck (2007).

have been regarded as a source of risk, and as a risk that receives a price in the market place.[39]

Our model, however, says that liquidity risk should be captured by the fluctuations in the shadow prices R, not by the empirical variable LIQ. We now ask whether the unanticipated component of LIQ is a good substitute for the right measure.[40] That question is answered in the new Figure 12, which shows correlations across sample paths between the two variables. We conclude that LIQ seems to only capture liquidity risk in a time-series dimension for small transactions costs values. For reasonable values of transactions costs, the correlation between these variables is close to zero, which means that LIQ is not an adequate proxy in tests of the CAPM.[41]

5 Conclusion

We have developed a new method to compute financial-market equilibria in the presence of proportional transactions costs. For a given rate of transactions costs, our method delivers the optimal, market-clearing moves of each investor and the resulting ticker and transactions prices.

We have concluded that transactions costs have a strong effect on investors' asset holdings, that deviations in asset prices from a frictionless economy are due to (i) current transactions costs and (ii) all future price differences due to state price differences, and that transactions costs may explain some parts of empirical asset pricing anomalies.

We have presented a transactions-costs adjusted CAPM model and identified the risk factors. We confirmed, however, the view expressed in prior work saying that explicitly observable transactions costs cannot account for the size of what is commonly measured as a liquidity premium. We have commented, in the light of our theoretical model, on the adequacy of extant empirical tests of CAPMs that include a premium for liquidity risk. Shadow prices that properly capture liquidity in the very definition of rates of return are generally not observable and the variables capturing liquidity in CAPM tests do not seem to be adequate proxies

If one wanted to test our CAPM, one would not use the standard concept of rate of return measured between fixed points in time. Instead, one would use transactions prices only, which do not occur at fixed time intervals, and one would substitute out the values of prices that are unobserved for lack of transaction. Further work is needed to develop the econometric method.

Other work should aim to model an equilibrium in which trading would not be Walrasian. In it, the rate of transactions costs would not be a given and

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[39] Acharya and Pedersen (2005) do use LIQ (or ILLIQ) as the liquidity measure but Pástor and Stambaugh (2003) use a different measure, which we cannot replicate here.

[40] We remind the reader that the separation between anticipated an unanticipated component is made by means of an autoregressive process of order two fitted on each sample path, as might have been done in empirical work.

[41] The behavior is similar for both agents, only with a reversed sign due to the restriction on the R variables.

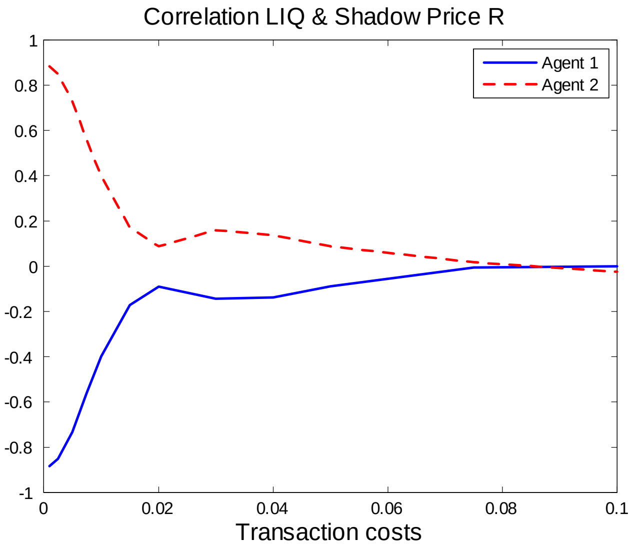

Figure 12: **Correlation between the LIQ variable and the shadow prices** R **of the two agents**. The computation uses results up to period 25 for different levels of transactions costs, in the range from 0% to 10%. All parameters and variables are set at their benchmark values indicate din table 1. We use 20,000 simulations along the tree. Recall that, if expressed in terms of value of trades rather than number of shares traded, the rate of transactions costs would be reduced by a factor of about 25, which is the approximate value of the share of stock.

investors would submit limit and market orders. The behavior of the limit-order book would be obtained. This would be similar to Foucault (1999) and Roşu (2009) and much of the Microstructure literature except that trades would arrive at the time of the investor's choice, not as the result of an exogenous Poisson process. Recently, Kühn and Stroh (2010) have used the dual approach to optimize portfolio choice in a limit-order market and may have shown the way to do that.

Appendixes

A Proof of the equation system of Section 1.

The Lagrangian for problem (3) is:

$$L_l\left(\{\theta_{l,t-1,i}\},\cdot,e_{l,t},t\right)=\sup_{c_{l,t},\left\{\widehat{\theta}_{l,t,i},\widehat{\widehat{\theta}}_{l,t,i}\right\}}\ \inf_{\phi_{l,t}} u_l\left(c_{l,t},t\right)$$

$$+\sum_{j=u,d}\pi_{t,t+1,j}J_l\left(\left\{\widehat{\theta}_{l,t,i}+\widehat{\widehat{\theta}}_{l,t,i}-\theta_{l,t-1,i}\right\},\cdot,e_{l,t+1,j},t+1\right)$$

$$+\phi_{l,t}\left[e_{l,t}+\sum_{i=1,2}\theta_{l,t-1,i}\delta_{t,i}-c_{l,t}\right.$$

$$\left.-\sum_{i=1,2}\left(\widehat{\theta}_{l,t,i}-\theta_{l,t-1,i}\right)\left(S_{t,i}+\lambda_{i,t}\right)-\sum_{i=1,2}\left(\widehat{\widehat{\theta}}_{l,t,i}-\theta_{l,t-1,i}\right)\left(S_{t,i}-\varepsilon_{i,t}\right)\right]$$

$$+\sum_{i}\left[\mu_{1,l,t,i}\left(\widehat{\theta}_{l,t,i}-\theta_{l,t-1,i}\right)+\mu_{2,l,t,i}\left(\theta_{l,t-1,i}-\widehat{\widehat{\theta}}_{l,t,i}\right)\right]$$

where $\phi_{l,t}$ is obviously the Lagrange multiplier attached to the flow budget constraint (4) and μ_1 and μ_2 are the Lagrange multipliers attached to the inequality constraints (5). The Karush-Kuhn-Tucker first-order conditions are:

$$u_l'\left(c_{l,t},t\right)=\phi_{l,t}$$

$$e_{l,t}+\sum_{i=1,2}\theta_{l,t-1,i}\delta_{t,i}-c_{l,t}-\sum_{i=1,2}\left(\widehat{\theta}_{l,t,i}-\theta_{l,t-1,i}\right)\left(S_{t,i}+\lambda_{i,t}\right)$$

$$-\sum_{i=1,2}\left(\widehat{\widehat{\theta}}_{l,t,i}-\theta_{l,t-1,i}\right)\left(S_{t,i}-\varepsilon_{i,t}\right)=0$$

$$\sum_{j=u,d}\pi_{t,t+1,j}\frac{\partial J_{l,t+1,j}}{\partial\theta_{l,t,i}}\left(\left\{\widehat{\theta}_{l,t,i}+\widehat{\widehat{\theta}}_{l,t,i}-\theta_{l,t-1,i}\right\},\cdot,e_{l,t+1,j},t+1\right)\tag{13}$$

$$=\phi_{l,t}\times\left(S_{t,i}+\lambda_{i,t}\right)-\mu_{1,l,t,i}$$

$$\sum_{j=u,d}\pi_{t,t+1,j}\frac{\partial J_{l,t+1,j}}{\partial\theta_{l,t,i}}\left(\left\{\widehat{\theta}_{l,t,i}+\widehat{\widehat{\theta}}_{l,t,i}-\theta_{l,t-1,i}\right\},\cdot,e_{l,t+1,j},t+1\right)$$

$$=\phi_{l,t}\times\left(S_{t,i}-\varepsilon_{i,t}\right)+\mu_{2,l,t,i}$$

$$\widehat{\widehat{\theta}}_{l,t,i}\leq\theta_{l,t-1,i}\leq\widehat{\theta}_{l,t,i};\mu_{1,l,t,i}\geq 0;\mu_{2,l,t,i}\geq 0$$

$$\mu_{1,l,t,i}\times\left(\widehat{\theta}_{l,t,i}-\theta_{l,t-1,i}\right)=0$$

$$\mu_{2,l,t,i}\times\left(\theta_{l,t-1,i}-\widehat{\widehat{\theta}}_{l,t,i}\right)=0$$

where the last equations are referred to as "the complementary-slackness" conditions. Two of the first-order conditions imply that

$$\phi_{l,t} \times (S_{t,i} + \lambda_{i,t}) - \mu_{1,l,t,i} = \phi_{l,t} \times (S_{t,i} - \varepsilon_{i,t}) + \mu_{2,l,t,i}$$

Therefore, we can merge two Lagrange multipliers into one, $R_{l,t,i}$, defined as:

$$\phi_{l,t} \times (R_{l,t,i} + S_{t,i}) \triangleq \phi_{l,t} \times (S_{t,i} + \lambda_{i,t}) - \mu_{1,l,t,i} = \phi_{l,t} \times (S_{t,i} - \varepsilon_{i,t}) + \mu_{2,l,t,i}$$

and recognize one first-order condition that replaces two of them:

$$\begin{aligned} &\sum_{j=u,d} \pi_{t,t+1,j} \frac{\partial J_{l,t+1,j}}{\partial \theta_{l,t,i}} \left(\left\{ \widehat{\theta}_{l,t,i} + \widehat{\widehat{\theta}}_{l,t,i} - \theta_{l,t-1,i} \right\}, \cdot, e_{l,t+1,j}, t+1 \right) \\ &\qquad = \phi_{l,t} \times (R_{l,t,i} + S_{t,i}) \end{aligned} \tag{14}$$

In order to eliminate the value function from the first-order conditions, we differentiate the Lagrangian with respect to $\theta_{l,t-1,i}$ and then make use of (14):

$$\begin{aligned} \frac{\partial L_l}{\partial \theta_{l,t-1,i}} &= \frac{\partial J_l}{\partial \theta_{l,t-1,i}} \\ &= -\sum_{j=u,d} \pi_{t,t+1,j} \frac{\partial J_{l,t+1,j}}{\partial \theta_{l,t,i}} \left(\left\{ \widehat{\theta}_{l,t,i} + \widehat{\widehat{\theta}}_{l,t,i} - \theta_{l,t-1,i} \right\}, \cdot, e_{l,t+1,j}, t+1 \right) \\ &\quad + \phi_{l,t} \left[\delta_{t,i} + (S_{t,i} + \lambda_{i,t}) + (S_{t,i} - \varepsilon_{i,t}) \right] - \mu_{1,l,t,i} + \mu_{2,l,t,i} \\ &= -\sum_{j=u,d} \pi_{t,t+1,j} \frac{\partial J_{l,t+1,j}}{\partial \theta_{l,t,i}} \left(\left\{ \widehat{\theta}_{l,t,i} + \widehat{\widehat{\theta}}_{l,t,i} - \theta_{l,t-1,i} \right\}, \cdot, e_{l,t+1,j}, t+1 \right) \\ &\quad + \phi_{l,t} \delta_{t,i} + 2\phi_{l,t} \times (R_{l,t,i} + S_{t,i}) \\ &= \phi_{l,t} \left(\delta_{t,i} + R_{l,t,i} + S_{t,i} \right) \end{aligned}$$

so that the first-order conditions can also be written:

$$\begin{gathered} u_l'(c_{l,t}, t) = \phi_{l,t}; l = 1,2 \\ e_{l,t} + \sum_{i=1,2} \theta_{l,t-1,i} \delta_{t,i} - c_{l,t} - \sum_{i=1,2} \left(\widehat{\theta}_{l,t,i} + \widehat{\widehat{\theta}}_{l,t,i} - 2 \times \theta_{l,t-1,i} \right) (R_{l,t,i} + S_{t,i}) = 0 \\ \sum_{j=u,d} \pi_{t,t+1,j} \times \phi_{l,t+1,j} \times (\delta_{t+1,i,j} + R_{l,t+1,i,j} + S_{t+1,i,j}) = \phi_{l,t} \times (R_{l,t,i} + S_{t,i}) \end{gathered} \tag{15}$$

$$\begin{gathered} \widehat{\widehat{\theta}}_{l,t,i} \leq \theta_{l,t-1,i} \leq \widehat{\theta}_{l,t,i} \\ -\varepsilon_{i,t} \leq R_{l,t,i} \leq \lambda_{i,t}; \\ (-R_{l,t,i} + \lambda_{i,t}) \times \left(\widehat{\theta}_{l,t,i} - \theta_{l,t-1,i} \right) = 0 \\ (R_{l,t,i} + \varepsilon_{i,t}) \times \left(\theta_{l,t-1,i} - \widehat{\widehat{\theta}}_{l,t,i} \right) = 0 \end{gathered}$$

As has been noted by Dumas and Lyasoff (2010) in a different context, the system made of (15) and (6) above has a drawback. It must be solved simultaneously (or globally) for all nodes of all times. As written, it cannot be solved recursively in the backward way because the unknowns at time t include consumptions at time t, $c_{l,t}$, whereas the third subset of equations in (15) if rewritten as:

$$\sum_{j=u,d} \pi_{t,t+1,j} \times u_l' \left(c_{l,t+1,j}, t\right) \times \left[\delta_{t+1,i,j} + R_{l,t+1,i,j} + S_{t+1,i,j}\right]$$
$$= \phi_{l,t} \times \left(R_{l,t,i} + S_{t,i}\right); l = 1, 2$$

can be seen to be a restriction on consumptions at time $t+1$, which at time t would already be solved for.

In order to "synchronize" the solution algorithm of the equations and allow recursivity, we first shift all first-order conditions except the third one forward and, second, we no longer make explicit use of the investor's positions $\theta_{l,t-1,i}$ held when entering time t, focusing instead on the positions $\theta_{l,t+1,i,j}$ ($\sum_{l=1,2} \theta_{l,t+1,i,j} = 0$) held when exiting time $t+1$. Regrouping equations in that way leads to the equation system of Section 1.

B Time 0

After solving the equation system of Section 1, it remains to solve at time 0 the following equation system ($t = -1, t+1 = 0$) *from which the kernel conditions only have been removed*:[42]

1. First-order conditions for time 0 consumption:

$$u_l' \left(c_{l,0}, 0\right) = \phi_{l,0}$$

2. The set of time-0 flow budget constraints for all investors and all states of nature of that time:

$$e_{l,0} + \sum_{i=1,2} \theta_{l,-1,i} \delta_{0,i} - c_{l,0}$$
$$- \sum_{i=1,2} \left(\theta_{l,0,i} - \theta_{l,-1,i}\right) \left(R_{l,0,i} + S_{0,i}\right) = 0$$

3. Definitions:

$$\theta_{l,0,i} = \widehat{\theta}_{l,0,i} + \widehat{\widehat{\theta}}_{l,0,i} - \theta_{l,-1,i}$$

---

[42] There could be several possible states j at time 0 but we have removed the subscript j.

4. Complementary-slackness conditions:

$$\begin{aligned}(-R_{l,0,i}+\lambda_{i,0})\times\left(\widehat{\theta}_{l,0,i}-\theta_{l,-1,i}\right) &= 0\\ (R_{l,0,i}+\varepsilon_{i,0})\times\left(\theta_{l,-1,i}-\widehat{\widehat{\theta}}_{l,0,i}\right) &= 0\end{aligned}$$

5. Market-clearing restrictions:

$$\sum_{l=1,2}\theta_{l,-1,i}=0 \text{ or } 1$$

This system can be handled in one of two ways:

1. We can either solve for the unknowns $\left\{c_{l,0},R_{l,0,i},\theta_{l,-1,i},\widehat{\theta}_{l,0,i},\widehat{\widehat{\theta}}_{l,0,i};l=1,2;j=u,d\right\}$ as functions of $\left\{\phi_{l,0}\right\}$ and $\{R_{l,0,i}\}$. If we plot $\theta_{l,-1,i}$ as functions of $\left\{\phi_{l,0}\right\}$ and $\{R_{l,0,i}\}$, we have the "Negishi map".[43] If it is invertible, we can then invert that Negishi map to obtain the values of $\left\{\phi_{l,0}\right\}$ and $\{R_{l,0,i}\}$ such that $\theta_{l,-1,i}=\bar{\theta}_{l,i}$. If the values $\bar{\theta}_{l,i}$ fall outside the image set of the Negishi map, there simply does not exist an equilibrium as one investor would, at equilibrium prices, be unable to repay his/her debt to the other investor.
2. Or *we drop the market-clearing equation also* and solve directly this system for the unknowns: $\left\{c_{l,0},\phi_{l,0},R_{l,0,i},\widehat{\theta}_{l,0,i},\widehat{\widehat{\theta}}_{l,0,i};l=1,2;j=u,d\right\}$ with $\theta_{l,-1,i}$, replaced in the system by the given $\bar{\theta}_{l,i}$.

In this paper, the second method has been used.

C Scale-invariance property

Assuming that the transactions costs per share traded are proportional to the economy's endowment e_1, we now show that all the nodes of a given point in time, which differ only by their value of the exogenous variable, are isomorphic to each other, where the isomorphy simply means that we can factor out the endowment. We call $\bar{\lambda}\triangleq\lambda/e_1$ and $\bar{\varepsilon}\triangleq\varepsilon/e_1$ the rescaled transactions costs and $\bar{R}_{l,t,i}\triangleq R_{l,t,i}/e_1$ the rescaled shadow prices.

Time T-1

Given the fact that we have zero transactions costs in the last period T, using the first-order conditions for consumption, and rewriting the investors'

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[43] For a definition of the "Negishi map" in a market with frictions, see Dumas and Lyasoff (2010).

consumptions in terms of consumption shares $\omega_{l,T,j}$ the system of equations at time $T-1$ can be re-written as:

$$e_{l,T,j} + \sum_{i=1,2} \theta_{l,T-1,i}\delta_{T,i,j} - \omega_{l,T,j} \times e_{l,T,j} = 0$$

$$\begin{aligned} & -\bar{R}_{1,T-1,i} \times e_{1,T-1} + \beta_1 \sum_{j=1,2} \pi_{T-1,T,j} \times \left(\frac{\omega_{1,T,j}}{\omega_{1,T-1}} \times \frac{e_{1,T,j}}{e_{1,T-1}} \right)^{-\gamma_1} \times \delta_{T,i,j} \\ = \quad & -\bar{R}_{2,T-1,i} \times e_{1,T-1} + \beta_2 \sum_{j=1,2} \pi_{T-1,T,j} \times \left(\frac{\omega_{2,T,j}}{\omega_{2,T-1}} \times \frac{e_{1,T,j}}{e_{1,T-1}} \right)^{-\gamma_2} \times \delta_{T,i,j} \end{aligned}$$

$$\sum_{l=1,2} \theta_{l,T-1,i} = 0$$

with unknowns $\{\omega_{l,T,j}; l = 1,2; j = 1,2\}$, $\{\theta_{l,T-1,i}; l = 1,2; i = 1,2\}$.

Using the fact that the risky asset pays as dividends the endowment of Investor 1, i.e. $\delta_{T,2,j} = e_{1,T,j}$,and that the riskless asset has a unit payoff, we can solve the flow budget equation for $j = 1$ for the holdings in the first asset:

$$\theta_{l,T-1,1} = e_{1,T-1} \times u \times (\omega_{l,T,1} - 1_{l,E} - \theta_{l,T-1,2}), \tag{16}$$

where u is the size of the multiplicative up move in the tree (d is the down move) and where $1_{l,E}$ denotes an indicator for receiving endowment, i.e. $1_{1,E} = 1$ and $1_{2,E} = 0$. Plugging this expression into the flow budget equation for $j = 2$, we can solve for $\theta_{l,T-1,2}$:

$$\theta_{l,T-1,2} = \frac{1_{1,E} \times (d-u) - \omega_{l,T,1} \times d + \omega_{l,T,2} \times u}{u-d}. \tag{17}$$

Rewriting the kernel conditions and reducing the system using (16) and (17), we get a system with unknowns $\{\omega_{l,T,j}; l = 1,2; j = 1,2\}$ only:

$$\beta_1 \sum_{j=u,d} \pi_{T-1,T,j} \left(\frac{\omega_{1,T,j}}{\omega_{1,T-1}} \right)^{-\gamma_1} r_j^{-\gamma_1} = \beta_2 \sum_{j=u,d} \pi_{T-1,T,j} \left(\frac{\omega_{2,T,j}}{\omega_{2,T-1}} \right)^{-\gamma_2} r_j^{-\gamma_2}$$

$$\begin{aligned} & -\bar{R}_{1,T-1,i} + \beta_1 \sum_{j=u,d} \pi_{T-1,T,j} \left(\frac{\omega_{1,T,j}}{\omega_{1,T-1}} \right)^{-\gamma_1} r_j^{-\gamma_1+1} \\ = \quad & -\bar{R}_{2,T-1,i} + \beta_2 \sum_{j=u,d} \pi_{T-1,T,j} \left(\frac{\omega_{2,T,j}}{\omega_{2,T-1}} \right)^{-\gamma_2} r_j^{-\gamma_2+1} \end{aligned}$$

$$\frac{\omega_{1,T,2} \times u - \omega_{1,T,1} \times d}{u-d} + \frac{\omega_{2,T,2} \times u - \omega_{2,T,1} \times d}{u-d} = 0$$

$$\left(\omega_{1,T,1} - \frac{\omega_{1,T,2} \times u - \omega_{1,T,1} \times d}{u-d} \right) + \left(\omega_{2,T,1} - \frac{\omega_{2,T,2} \times u - \omega_{2,T,1} \times d}{u-d} \right) = 0$$

where $r_j = u$ for $j = 1$ and $r_j = d$ for $j = 2$.

Importantly this system of equations *does not depend on the current or future levels of endowment*, i.e. it is enough to solve the system for one node at time $T-1$ as long as u and d are not state (node) dependent.

After solving this system, one can compute the implied holdings and asset prices. From (17) we get that the stock holdings are independent of $T-1$ endowment, while from (16) we know that the bond holdings are scaled by the $T-1$ endowment:

$$\begin{aligned}\theta_{l,T-1,1} &= e_{1,T-1} \times u \times \left(\omega_{l,T,1} - 1_{1,E} - \frac{1_{1,E}\,(d-u) - \omega_{l,T,1} d + \omega_{l,T,2} u}{u-d}\right) \\ &= e_{1,T-1} \times \bar{\theta}_{l,T-1,1}, \qquad (18)\end{aligned}$$

where $\bar{\theta}_{l,T-1,1}$ denotes the normalized bond holdings for $e_{1,T-1} = 1$. Moreover, we get that the bond price does not depend on $T-1$ endowment:

$$S_{T-1,1} = \beta_1 \sum_{j=u,d} \pi_{T-1,T,j} \times \left(\frac{\omega_{1,T,j}}{\omega_{1,T-1}}\right)^{-\gamma_1} \times r_j^{-\gamma_1},$$

and that the stock price is scaled by the $T-1$ endowment:

$$\begin{aligned}S_{T-1,2} &= e_{1,T-1} \times \left[-\bar{R}_{1,T-1,i} + \beta_1 \sum_{j=u,d} \pi_{T-1,T,j} \times \left(\frac{\omega_{1,T,j}}{\omega_{1,T-1}}\right)^{-\gamma_1} \times r_j^{-\gamma_1}\right] \\ &\triangleq e_{1,T-1} \times \bar{S}_{T-1,2}, \qquad (19)\end{aligned}$$

where $\bar{S}_{T-1,2}$ denotes the normalized price for $e_{1,T-1} = 1$.

Time t<T-1

For time $t < T-1$ the system of equations is the system of Section 1. Rewriting $c_{l,t+1,j} = \omega_{l,t+1,j} \times e_{1,t+1,j}$, replacing $S_{t+1,2}$ and $\theta_{l,T-1,1}$ with expressions (19) and (18), and solving the flow budget equation for $j = 1$ for $\theta_{l,t,1}$, we get:

$$\theta_{l,t,1} = e_{1,t} \times u \times \left[\begin{array}{c} \omega_{l,t+1,1} + (\theta_{l,t+1,2,j} - \theta_{l,t,2})\left(\bar{R}_{l,t+1,2,1} + \bar{S}_{t+1,2,1}\right) \\ +\bar{\theta}_{l,T-1,1} S_{t+1,1,1} - 1_{l,E} - \theta_{l,t,2} \end{array}\right]$$

Plugging this into the budget equation for $j = 2$, and solving for $\theta^S_{l,t}$ we get:

$$\begin{aligned}\theta_{l,t,2} = {} & \frac{1}{\left(d \times \left(\bar{R}_{l,t+1,2,2} + \bar{S}_{T-1,2,2}\right) - u \times \left(\bar{R}_{l,t+1,2,1} + \bar{S}_{T-1,2,1}\right)\right)} \times \qquad (20) \\ & [d \times \left(\omega_{l,t+1,2} - 1_{l,E} + \bar{\theta}_{l,t+1,1,2} S_{t+1,1,2} + \theta_{l,t+1,2,2}\left(\bar{R}_{l,t+1,2,2} + \bar{S}_{t+1,2,2}\right)\right) \\ & -u \times \left(\omega_{l,t+1,1} - 1_{l,E} + \bar{\theta}_{l,t+1,1,1} S_{t+1,1,1} + \theta_{l,t+1,2,1}\left(\bar{R}_{l,t+1,2,1} + \bar{S}_{t+1,2,1}\right)\right)]\end{aligned}$$

Rewriting the kernel conditions, we can write the system as:

$$\beta_1 \sum_{j=u,d} \pi_{T-1,T,j} \left(\frac{\omega_{1,T,j}}{\omega_{1,T-1}}\right)^{-\gamma_1} r_j^{-\gamma_1} = \beta_2 \sum_{j=u,d} \pi_{T-1,T,j} \left(\frac{\omega_{2,T,j}}{\omega_{2,T-1}}\right)^{-\gamma_2} r_j^{-\gamma_2}$$

$$
\begin{aligned}
& -\bar{R}_{1,t,i} + \beta_1 \sum_{j=u,d} \pi_{t,t+1,j} \left(\frac{\omega_{1,t+1,j}}{\omega_{1,t}} \right)^{-\gamma_1} r_j^{-\gamma_1+1} \left(1 + \bar{R}_{1,t+1,i,j} + \bar{S}_{t+1,2,j} \right) \\
= & -\bar{R}_{2,t,i} + \beta_2 \sum_{j=u,d} \pi_{t,t+1,j} \left(\frac{\omega_{2,t+1,j}}{\omega_{2,t}} \right)^{-\gamma_2} r_j^{-\gamma_2+1} \left(1 + \bar{R}_{2,t+1,i,j} + \bar{S}_{t+1,2,j} \right)
\end{aligned}
$$

$$
\theta_{l,t+1,i,j} = \widehat{\theta}_{l,t+1,i,j} + \widehat{\widehat{\theta}}_{l,t+1,i,j} - \theta_{l,t,i}
$$

$$
\begin{aligned}
\left(-\bar{R}_{l,t+1,i,j} + \bar{\lambda}_{i,t+1,j} \right) \times \left(\widehat{\theta}_{l,t+1,i,j} - \theta_{l,t,i} \right) &= 0 \\
\left(\bar{R}_{l,t+1,i,j} + \bar{\varepsilon}_{i,t+1,j} \right) \times \left(\theta_{l,t,i} - \widehat{\widehat{\theta}}_{l,t+1,i,j} \right) &= 0
\end{aligned}
$$

$$
\sum_{l=1,2} \theta_{l,t,i} = 0
$$

with unknowns $\left\{ \omega_{l,t+1,j}; \bar{R}_{l,t+1,j}; \widehat{\theta}_{l,t+1,i,j}; \widehat{\widehat{\theta}}_{l,t+1,i,j}; l = 1,2; j = 1,2 \right\}$. The holdings implied are given by (20) and an analogous equation for the bond. Note, one can show that the endowment $e_{1,t}$ cancels out in the market clearing conditions. This system *does not depend on the level of endowment* $e_{1,t}$, only on u as well as d, and therefore we only need to solve the system at one node at time t.

As backward interpolated values we use the bond price $S_{t+1,2,j}$ and stock holdings $\theta_{l,t+1,2,j}$ as well as the normalized stock price $\bar{S}_{t+1,2,j}$ and normalized bond holdings $\bar{\theta}_{l,t+1,1,j}$. After solving the system we can compute the implied time t holdings and prices. Again, holdings in the bond and the stock price are scaled by $e_{1,t}$, while the holdings in the stock and the bond price are not scaled. *Using backward induction the scaling invariance holds for any time* t.

D Proof of Proposition 3

The proof is by induction.

At date $t = T - 1$, the present value of dividends δ at time $T - 1$ is given by:

$$
\hat{S}_{T-1,i,l} = \mathbb{E}_{T-1} \left[\frac{\phi_{l,T}}{\phi_{l,T-1}} \times \delta_{T,i} \right]
$$

The stock price is given by:

$$
\begin{aligned}
S_{T-1,i} &= -R_{l,T-1,i} + \mathbb{E}_{T-1} \left[\frac{\phi_{l,T}}{\phi_{l,T-1}} \times \delta_{T,i} \right] \\
&= -R_{l,T-1,i} + \hat{S}_{T-1,i,l} \qquad (21)
\end{aligned}
$$

At $t = T - 2$, the present value of dividends is:

$$
\hat{S}_{T-2,i,l} = \mathbb{E}_{T-2} \left[\frac{\phi_{l,T-1}}{\phi_{l,T-2}} \times \left(\delta_{T,i} + \hat{S}_{T-1,i,l} \right) \right]
$$

while the stock price is:

$$\begin{aligned}
S_{T-2,i} &= -R_{l,T-2,i} + \mathbb{E}_{T-2}\left[\frac{\phi_{l,T-1}}{\phi_{l,T-2}} \times (\delta_{T-1,i} + R_{l,T-1,i} + S_{T-1,i})\right] \\
&= -R_{l,T-2,i} + \mathbb{E}_{T-2}\left[\frac{\phi_{l,T-1}}{\phi_{l,T-2}} \times \left(\delta_{T-1,i} + \hat{S}_{T-1,i,l}\right)\right] \\
&= -R_{l,T-2,i} + \hat{S}_{T-2,i,l}
\end{aligned}$$

where we used equation (21) to replace $S_{T-1,i}$. etc.

E Proof of Proposition 4

Time $T-1$

The stock price in an economy without transactions costs is given by:

$$S^*_{T-1} = \mathbb{E}_{T-1}\left[\frac{\phi^*_{l,T}}{\phi^*_{l,T-1}}\delta_T\right]$$

whereas Equation (8) applied to time $T-1$ is:

$$S_{T-1} = -R_{l,T-1} + \mathbb{E}_{T-1}\left[\frac{\phi_{l,T}}{\phi_{l,T-1}}\delta_T\right]$$

which can be rewritten as:

$$\begin{aligned}
S_{T-1} &= -R_{l,T-1} + \mathbb{E}_{T-1}\left[\frac{\phi^*_{l,T}}{\phi^*_{l,T-1}}\delta_T\right] + \mathbb{E}_{T-1}\left[\left(\frac{\phi_{l,T}}{\phi_{l,T-1}} - \frac{\phi^*_{l,T}}{\phi^*_{l,T-1}}\right)\delta_T\right] \\
&= -R_{l,T-1} + \mathbb{E}_{T-1}\left[\frac{\phi^*_{l,T}}{\phi^*_{l,T-1}}\delta_T\right] + \mathbb{E}_{T-1}\left[\Delta\phi_{l,T}\delta_T\right]
\end{aligned}$$

where we defined:

$$\Delta\phi_{l,T} \triangleq \frac{\phi_{l,T}}{\phi_{l,T-1}} - \frac{\phi^*_{l,T}}{\phi^*_{l,T-1}}$$

The difference between the stock price in a zero-transactions costs economy and an economy with transactions costs is given by:

$$S_{T-1} - S^*_{T-1} = -R_{l,T-1} + \mathbb{E}_{T-1}\left[\Delta\phi_{l,T}\delta_T\right] \tag{22}$$

Time $T-2$

The stock price in an economy without transactions costs is given by:

$$S^*_{T-2} = \mathbb{E}_{T-2}\left[\frac{\phi^*_{l,T-1}}{\phi^*_{l,T-2}}\left(\delta_{T-1} + S^*_{T-1}\right)\right]$$

whereas Equation (8) applied to time $T-2$ is:

$$S_{T-2} = -R_{l,T-2} + \mathbb{E}_{T-2}\left[\frac{\phi_{l,T-1}}{\phi_{l,T-2}}\left(\delta_{T-1} + S_{T-1} + R_{l,T-1}\right)\right]$$

Replacing $S_{T-1} + R_{l,T-1}$ with expression (22), this can be rewritten as:

$$\begin{aligned}
S_{T-2} &= -R_{l,T-2} + \mathbb{E}_{T-2}\left[\frac{\phi_{l,T-1}}{\phi_{l,T-2}}\left(\delta_{T-1} + S^*_{T-1} + \mathbb{E}_{T-1}\left[\Delta\phi_{l,T}\delta_T\right]\right)\right] \\
&= -R_{l,T-2} + \mathbb{E}_{T-2}\left[\frac{\phi^*_{l,T-1}}{\phi^*_{l,T-2}}\left(\delta_{T-1} + S^*_{T-1}\right)\right] \\
&\quad + \mathbb{E}_{T-2}\left[\Delta\phi_{l,T-1}\left(\delta_{T-1} + S^*_{T-1}\right)\right] + \mathbb{E}_{T-2}\left[\frac{\phi_{l,T-1}}{\phi_{l,T-2}}\Delta\phi_{l,T}\delta_T\right] \\
&= -R_{l,T-2} + S^*_{T-2} \\
&\quad + \mathbb{E}_{T-2}\left[\Delta\phi_{l,T-1}\left(\delta_{T-1} + S^*_{T-1}\right)\right] + \mathbb{E}_{T-2}\left[\frac{\phi_{l,T-1}}{\phi_{l,T-2}}\Delta\phi_{l,T}\delta_T\right]
\end{aligned}$$

The difference between the stock price in a zero-transactions costs economy and an economy with transactions costs is given by:

$$S_{T-2} - S^*_{T-2} = -R_{l,T-2} + \mathbb{E}_{T-2}\left[\Delta\phi_{l,T-1}\left(\delta_{T-1} + S^*_{T-1}\right)\right] + \mathbb{E}_{T-2}\left[\frac{\phi_{l,T-1}}{\phi_{l,T-2}}\Delta\phi_{l,T}\delta_T\right]$$

Time t

By an induction argument one can show the final result (4).

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